Mathematics 255—February 12, 1996

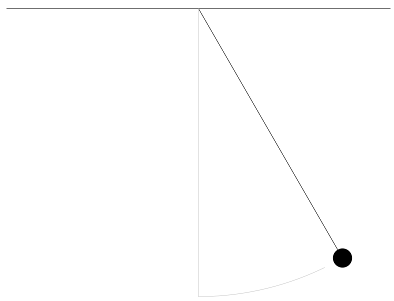
Impulse

The pendulum

Suppose a pendulum to be hanging free on a rigid arm. Let

$$\begin{aligned} \ell &= \text{its length} \\ m &= \text{the mass of the bob at the end} \\ y &= \text{the displacement of the bob from the vertical position, measured along the arc} \\ \theta &= \text{the angular displacement: } y = \ell\theta \\ c &= \text{a friction coefficient} \\ F(t) &= \text{external force} \end{aligned}$$

The external force, for example, might be one caused by shaking the top of the pendulum back and forth.



The component of gravity exerted along the arc of the path is $-mg\sin\theta$. The friction force exerted is assumed to be proportional to velocity. Newton's law here asserts that

$$my'' = -cy' - mg\,\sin\theta + F(t)$$

or

$$my'' + cy' + mg\,\sin(y/\ell) = F(t)\ .$$

We shall look at this equation later on in detail, but for the moment we are going to consider only motions of a large massive pendulum which does not move far from the vertical position. In this situation, θ remains small and $\sin\theta$ is approximately the same as θ. So we get as an approximation the equation

$$my'' + cy' + ky = F(t)$$

where $k = mg/\ell$. This is a linear equation, and it is called the **linear approximation** of the original one, for motions in the neighbourhood of the equilibrium position at dead vertical.

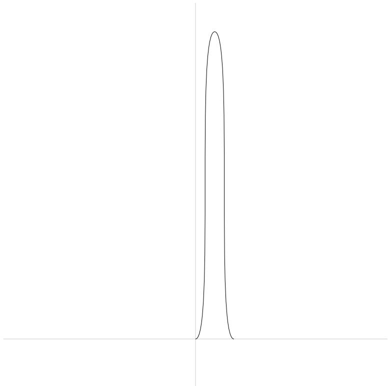
Impulse and inhomogeneous equations

Suppose the pendulum sits at rest until $t = 0$, when it is hit sharply with a hammer, say in the direction of positive displacement (from the left, in the picture above). What determines how the pendulum moves? The details are messy, of course: when the head hits the bob, the left hand side of the bob squishes in a bit, reacting elastically against the head of the hammer but also transmitting the effect of the blow throughout the bob, eventually to the rod of the pendulum, and then up the rod. But in a short time things stabilize and the pendulum begins to swing back and forth in a simple fashion. After this happens, the force $F(t)$ is equal to 0 and the system obeys the homogeneous equation

$$my'' + cy' + ky = 0$$

The period when interesting things are going on will last a fraction of a second. At the end of that short period, where is the pendulum, and how fast is it moving? The answers to this question will determine the motion from that point on, since we just solve the homogeneous equation with appropriate initial conditions.

Let's answer the second question first: *how fast is the bob moving when the hammer stops contact?* Of course this depends on how the force acts. Here, for example, is a plausible graph of $F(t)$:



What we shall see is that the effect of the force depends in a very simple way on this graph, at least if the time tinterval Δt is small. Of course the effect of the force is mainly to determine the position and velocity at the end of the initial period. The claim is certainly true as ar as position is concerned: if the time interval is small then the position of the bob won't have changed at all. As far as velocity is concerned, we can use Newton's law $F = ma$, which can be reformulated approximately as

$$F = m\,\Delta v/\Delta t, \quad F\,\Delta t = m\,\Delta v$$

which means that roughly speaking *the change in momentum is the area under the graph of $F(t)$*, which is the integral of $F(t)\,dt$. We are assuming that the effects of the force of gravity are small over the small interval, compared to the force F. This is not badly inaccurate.

So the basic idea is that the effect of the hammer blow on velocity depends primarily on the area under its graph, and it is in fact proportional to this area. I repeat that the effect on position is a second-order effect, which is to say it shrinks proportionally to the square of the length of the time interval in which things are going on. This is just what happens when you let an object drop from your hand, and the length of the drop is proportional to t^2.

The integral of $F dt$ is called **impulse** or **action**. It measures the transfer of momentum. An **ideal** hammer would transfer the momentum instantaneously, and we can measure even such ideal hammers according to the amount of momentum transferred. The ideal **unit hammer** can be thought of as a limit of ones acting over a very small but still finite interval, each with unit area under the graph of $F(t)$. We can now ask a simpler equation than the original one: *What is the effect of a a unit ideal hammer?* It changes the momentum of the pendulum from 0 to 1 in an instant. After that instant, the pendulum motion follows the homogeneous differential equation. So up to $t = 0$ the graph of position is dead horizontal, and at $t = 0$ it starts to rise suddenly with a slope equal to $1/m$. In other words, for $t > 0$ the position is given by the solution $y(t)$ of the homogeneous equation with $y(0) = 0$, $y'(0) = 1/m$. *This is what I have called, in the case where $m = 1$, the second fundamental solution $y_{2,0}(t)$ of the homogeneous equation.* So we see that *there is a simple interpretation of this solution in terms of a transfer of unit momentum to a physical system at $t = 0$.*

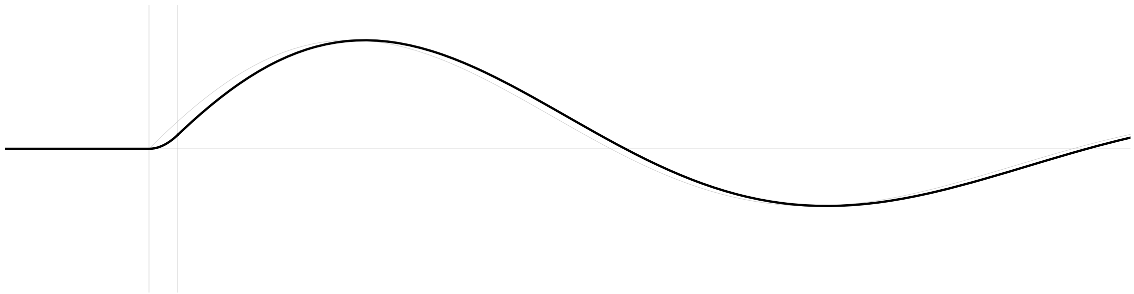
Here is the graph of the solution of the equation

$$y'' + 0.4y' + y = F(t), \quad y(0) = 0,\ y'(0) = 0$$

where

$$F(t) = \begin{cases} 5 & \text{if } 0 \le t \le 1/5 \\ 0 & \text{otherwise .} \end{cases}$$

I have chosen the friction coefficient low enough to allow some oscillation, and chosen $\Delta t = 1/5$ large enough to show what happens in that small interval. The light gray line is the actual function $y_{2,0}(t)$.



The physical significance of the formula

$$y(t) = \frac{1}{m} \int_0^t y_{\#,0}(t - s) F(s)\, dt$$

for the solution of the inhomogeneous system with vanishing initial conditions is that the effect is the same as a limit of a succession of hammer blows over short intervals, each one contributing a certain amount of momentum to the system, which then accumulates.