Mathematics 255—February 15, 1996

Second order inhomogeneous equations, part II

The basic formula for solving inhomogeneous linear equations in terms of the solutions of the associated homogeneous equation has the virtue of working under just about all circumstances, but it is more complicated to use than necessary in most circumstances.

The structure of the formula

The formula as I have stated it gives the solution for the equation

$$y'' + a(t)y' + b(t)y = f(t)$$

with initial conditions

$$y(u) = y_0$$
$$y'(u) = v_0$$

It says that

$$y = y_0 \cdot y_{[1,0],u}(t) + v_0 \cdot y_{[0,1],u}(t) + \int_u^t y_{[0,1],s}(t) f(s)\, ds$$

The first component of the formula is

$$y_0 \cdot y_{[1,0],u}(t) + v_0 \cdot y_{[0,1],u}(t)$$

is a linear combination of the fundamental solutions $y_{[1,0],u}(t)$ and $y_{[0,1],u}(t)$ of the homogeneous equation, and is consequently itself a solution to the homogeneous equation. Of course its derivative is

$$y_0\, y'_{[1,0],u}(t) + v_0\, y'_{[0,1],u}(t)$$

If you set $t = u$ you will see, by definition of the fundamental solutions, that its value at u is y_0 and the value of its derivative is v_0.

The first term in the basic formula can be characterized as the unique solution y of the homogeneous equation with $y(u) = y_0$ and $y'(t) = v_0$. The formula

$$y = y_0\, y_{[1,0],u}(t) + v_0\, y_{[0,1],u}(t)$$

for it is certainly correct, but you should be aware that there may be more efficient ways to find it.

The second component of the basic formula is

$$\int_u^t y_{[0,1],s}(t) f(s)\, ds$$

This is the most important and interesting of the two components. It depends linearly on $f(t)$, and in particular vanishes if the equation is homogeneous. *The second component of the basic formula is the solution y of the inhomogeneous equation with $y(u) = 0$, $y'(u) = 0$.* I remind you that, in case the coefficients $a(t)$ and $b(t)$ are constants, in order to find $y_{[0,1],s}$ we use the translation principle to write

$$y_{[0,1],s}(t) = y_{[0,1],0}(t - s)$$

If we want to know the general solution of the inhomogeneous equation, without initial conditions specified, we may replace the first component by the general solution of the homogeneous equation and the second component by an indefinite integral

$$\int^t y_{[0,1],s}(t) f(s)\, ds$$

The point here is that we can find the general solution to the inhomogeneous equation by adding the solution of the homogeneous equation to any one solution of the inhomogeneous we can find. Sometimes there is a solution of the inhomogeneous equation much simpler than the rest.

When the right hand side is a simple exponential function

Suppose we want to find the general solution of the equation

$$y'' + 3y' + 2y = e^{ct}$$

where c is an unspecified constant.

The general solution to the homogeneous equation is

$$y = Ae^{-t} + Be^{-2t}$$

with

$$y' = -Ae^{-t} - 2Be^{-2t}$$

To find $y_{[0,1],0}$ we solve

$$\begin{aligned} A + B &= 0 \\ -A - 2B &= 1 \end{aligned}$$

to find

$$A = 1, \quad B = -1$$

and

$$\begin{aligned} y_{[0,1],0}(t) &= e^{-t} - e^{-2t} \\ y_{[0,1],s}(t) &= e^{-(t-s)} - e^{-2(t-s)} \\ &= e^{-t}e^{s} - e^{-2t}e^{2s} \end{aligned}$$

Therefore the second component of the basic formula (with indefinite integrals) is

$$\begin{aligned} \int^t y_{[0,1],s}(t)f(t)\,ds &= \int^t \left[e^{-t}e^{s} - e^{-2t}e^{2s}\right]e^{cs}\,ds \\ &= e^{-t}\int^t e^{(c+1)s}\,ds - e^{-2t}\int^t e^{(c+2)s}\,ds \\ &= e^{-t}\left[\frac{e^{(c+1)s}}{(c+1)}\right] - e^{-2t}\left[\frac{e^{(c+2)t}}{(c+2)}\right] \\ &= \frac{e^{ct}}{(c+1)} - \frac{e^{ct}}{(c+2)} \\ &= \frac{e^{ct}}{(c^2+3c+2)} \end{aligned}$$

This procedure doesn't work if $c + 1 = 0$ or $c + 2 = 0$.

These calculations suggest what happens in general.

Theorem. *If c is not a root of the characteristic equation*

$$c^2 + ac + b = 0$$

then for the inhomogeneous linear differential equation with constant coefficients

$$y'' + ay' + y = e^{ct}$$

the function

$$y = \frac{e^{ct}}{c^2 + ac + b}$$

is a solution.

This works in circumstances more varied than you might think. For example, to solve

$$y'' + 3y' + 2y = \cos t$$

we recall that $\cos t$ is the real part of e^{it}. Therefore we first solve

$$y'' + 3y' + 2y = e^{it}$$

We get

$$\begin{aligned} y &= \frac{e^{it}}{i^2 + 3i + 2} \\ &= \frac{e^{it}}{1 + 3i} \\ &= \frac{1 - 3i}{1 - 3i}\,\frac{e^{it}}{1 + 3i} \\ &= \frac{(1 - 3i)(\cos t + i \sin t)}{10} \\ &= \frac{(\cos t + 3 \sin t) + i(-3 \cos t + \sin t)}{10} \end{aligned}$$

whose real part, the solution we rae looking for, is

$$\frac{\cos t + 3 \sin t}{10}$$

There are various rules for what happens when c is a root of the characteristic equation, but probably the most straightforward thing to do in this exceptional circumstance is go back to the basic formula.

There is one case which is worth while pointing out, however. Suppose we are solving

$$y'' + \omega_0^2 y = \cos \omega t$$

The solutions to the homogeneous are

$$e^{\pm i \omega_0 t}$$

and the second fundamental solution is

$$\frac{\sin \omega_0 t}{\omega_0}$$

If $\omega \neq \omega_0$ we get the solution

$$\frac{\cos \omega t}{-\omega^2 + \omega_0^2}$$

but when $\omega = \omega_0$ we have to calculate

$$\begin{aligned} y &= \frac{1}{\omega} \int^t \sin \omega (t - s) \cos \omega s \, ds \\ &= \frac{\sin \omega t}{\omega} \int^t \cos^2 \omega s \, ds - \frac{\cos \omega t}{\omega} \int^t \sin \omega s \, \cos \omega s \, ds \\ &= \frac{\sin \omega t}{\omega} \int^t \frac{1 + \cos 2s}{2} \, ds - \frac{\cos \omega t}{\omega} \int^t \frac{\sin 2s}{2} \, ds \\ &= \frac{t \sin \omega t}{2\omega} + \frac{\sin \omega t \sin 2\omega t}{4\omega^2} - \frac{cos \omega t \cos 2\omega t}{4\omega^2} \\ &= \frac{t \sin \omega t}{2\omega} - \frac{\cos (2t - t)\omega}{4\omega^2} \\ &= \frac{t \sin \omega t}{2\omega} - \frac{\cos \omega t}{4\omega^2} \end{aligned}$$

Recall that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos 2s = 2\cos^2 s - 1$$
$$\sin 2s = 2\sin s \cos s$$

Also note that the second term $\cos \omega t / 4\omega^2$ in the expression above is a solution to the homogeneous linear equation, and can be absorbed into the other component. So we discover in the end that the solution to the equation

$$y'' + \omega^2 y = \cos \omega t$$

(the special case when $c = i\omega$ is a characteristic root) is equal to

$$a \cos \omega t + b \sin \omega t + \frac{t \sin \omega t}{2\omega}$$

The term

$$\frac{t \sin \omega t}{2\omega}$$

happens to be the solution $y(t)$ of the inhomogeneous equation with $y(0) = 0$, $y'(0) = 0$.