Mathematics 255—February 12, 1996

Second order inhomogeneous equations

There is a single formula for solving second order inhomogeneous linear equations, somewhat analogous to the formula for first order equations. Using it is not usually the best way to find solutions, but occasionally it is necessary. Stating the formula requires some preliminary discussion of homogeneous equations.

Fundamental solutions

If

$$y'' + a(t)y' + b(t)y = 0$$

is any second order linear homogeneous differential equation, then the **fundamental solutions** associated to the point $t = s$ are two solutions satisfying particularly simple initial conditions at s. The first fundamental solution is the solution $y(t) = y_{[1,0],s}(t)$ such that

$$\begin{aligned} y(s) &= 1 \\ y'(s) &= 0 \end{aligned}$$

and the second is the solution $y(t) = y_{[0,1],s}(t)$ such that

$$\begin{aligned} y(s) &= 0 \\ y'(s) &= 1 \end{aligned}$$

Be sure to keep in mind that these solutions depend on the parameter s. Keep also in mind that finding them may not be so easy, since it amounts to finding essentially all the solutions of the homogeneous equation. In practice we can do it only if a and b are constants.

One reason the fundamental solutions are useful is that if you want to solve the equation above with initial conditions

$$y(s) = y_0, \quad y'(s) = v_0$$

then you just set

$$y = y_0 \cdot y_{[1,0],s} + v_0 \cdot y_{[0,1],s} \ .$$

But usually it is more work to find $y_{[1,0],s}$ and $y_{[0,1],s}$ than to solve for the initial conditions directly.

Examples

(1) Take

$$y'' - y = 0, \quad s = 0 \ .$$

All solutions are combinations of e^t and e^{-t}, so we set

$$y = c_1 e^t + c_2 e^{-t}$$

and solve first the system

$$\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= c_1 - c_2 = 0 \end{aligned}$$

to get

$$y_{[1,0],0} = (1/2)(e^t + e^{-t})$$

Next we solve

$$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= c_1 - c_2 = 1 \end{aligned}$$

to get

$$y_{[0,1],0} = (1/2)(e^t - e^{-t})$$

(2) For

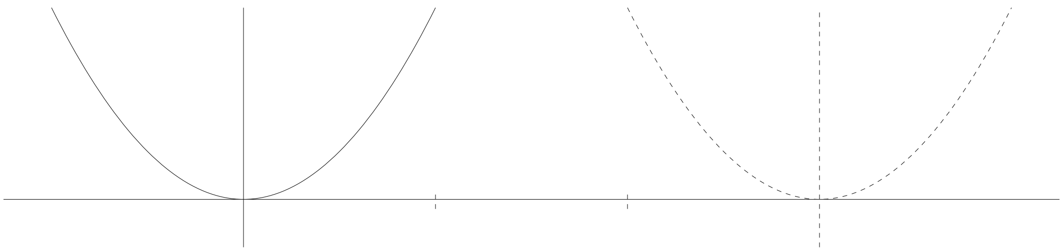
$$y'' + y = 0, \quad s = 0$$

we have

$$y_{[1,0],0} = \cos t$$
$$y_{[0,1],0} = \sin t$$

The translation principle

The graph of the function $y(t)$ when shifted to the right by s is the graph of the function $y(t-s)$. This new function of t is called the **translation** of y by s. Just below, for example, are the graphs of $y = t^2$ and $y = (t-3)^2$.



If $y(t)$ is a solution of the differential equation

$$y''(t) + a(t)y'(t) + b(t)y(t) = 0$$

Then also for any t

$$y''(t-s) + a(t-s)y'(t-s) + b(t-s)y(t-s) = 0 \; .$$

In particular, if $a(t)$ and $b(t)$ are constants we derive the **translation principle** for differential equations with constant coefficients. I have also called this the **principle of time invariance** since it amounts to saying that a physical system does not change in time. I recall that what it says is that *if $y(t)$ is a solution of a differential equation with constant coefficients then so are all translations $y(t-s)$.*

If $z(t) = y(t-s)$ then $z(s) = y(0)$ and $z'(s) = y'(0)$. Therefore for differential equations with constant coefficients

$$y_{[1,0],s} = y_{[1,0],0}(t-s)$$
$$y_{[0,1],s} = y_{[0,1],0}(t-s)$$

In other words, to find the fundamental solutions for any point s we have only to find those for $s = 0$ and translate them.

Wronskians

Suppose we have found a pair of independant solutions of a second order differential equation, say z_1 and z_2. If we want to find the fundamental solutions for a point $x = s$ then we set $y = c_1 z_1 + c_2 z_2$ and solve

$$\begin{aligned} c_1 z_1(s) + c_2 z_2(s) &= 1 \\ c_1 z_1'(s) + c_2 z_2'(s) &= 0 \end{aligned}$$

to get $y_{[1,0],s}$, and set $y = d_1 z_1 + d_2 z_2$ and solve

$$\begin{aligned} d_1 z_1(s) + d_2 z_2(s) &= 0 \\ d_1 z_1'(s) + d_2 z_2'(s) &= 1 \end{aligned}$$

to get $y_{[0,1],s}$. These equations say that

$$\begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$$

is the matrix inverse to

$$W(s) = \begin{bmatrix} z_1(s) & z_2(s) \\ z_1'(s) & z_2'(s) \end{bmatrix}$$

which is called the **Wronskian matrix** at s. Since this is only a 2×2 matrix we know that the inverse is

$$W^{-1}(s) = \frac{1}{\Delta} \begin{bmatrix} z_2'(s) & -z_2(s) \\ -z_1'(s) & z_1(s) \end{bmatrix}$$

where

$$\Delta = z_1(s) z_2'(s) - z_2(s) z_1'(s)$$

is called the **Wronskian determinant**.

In short we have for the second fundamental solution associated to s

$$\begin{aligned} y_{[1,0],s} &= c_1 z_1 + c_2 z_2 \\ y_{[0,1],s} &= d_1 z_1 + d_2 z_2 \end{aligned}$$

where

$$\begin{aligned} c_1 &= z_2'(s)/\Delta \\ c_2 &= -z_1'(s)/\Delta \\ d_1 &= -z_2(s)/\Delta \\ d_2 &= z_1(s)/\Delta \end{aligned}$$

A formula for solving second order inhomogeneous equations

I will now lay out a formula for solving the inhomogeneous equation and initial conditions

$$y'' + a(t)y' + b(t)y = c(t), \quad y(s) = y_0,\ y'(s) = v_0$$

given that we are able to find two independent solutions of the associated homogeneous equation. They are not easy to find in general, but I remind you that we can do it if the equation has constant coefficients.

In this formula the second fundamental solution $y_{[0,1],s}$ plays an important role. Write it now as

$$y_{\#,s} = y_{[0,1],s}$$

Step (1) Find the fundamental solutions $y_{[0,1],s}$, $y_{[0,1],s}$ and a formula for all $y_{\#,u} = y_{[0,1],u}$. Obtain these by means of the Wronskian formulas in the previous section or by translation from $y_{[1,0],0}$ and $y_{[0,1],0}$.

Step (2) The formula we want is

$$y(t) = y_0 \cdot y_{[1,0],s}(t) + v_0 \cdot y_{[0,1],s}(t) + \int_s^t y_{\#,u}(t)c(u)\,du$$

Example. If we want to solve

$$y'' + y = c(t), \quad y(0) = 2, \quad y'(0) = -1$$

we have

$$\begin{aligned} y_0 &= 2 \\ v_0 &= -1 \\ y_{\#,u} &= \sin(t-u) \\ y &= 2\cos t - \sin t + \int_0^t \sin(t-u)c(u)\,dt \\ &= 2\cos t - \sin t + \int_0^t (\sin t\cos u - \cos t \sin u)c(u)\,du \\ &= 2\cos t - \sin t + \sin t\int_0^t c(u)\,\cos u\,du - \cos t\int_0^t c(u)\,\sin u\,du \end{aligned}$$

If you put the formula in this section together with the formulas for $y_{[1,0],s}$ and $y_{[0,1],s}$ in the section on Wronskians, you will see that the formula here is exactly the same as the one the text *Boyce & DiPrima* derives in §3.7 by the method of **variation of parameters**. (Look at Theorem 3.7.1 on p. 167 as well as Problems # 22 and # 23 on p. 169.)

Analogies with the formula for first order linear equations

The formula for solving

$$y' + a(t)y = b(t), \quad y(s) = y_0$$

is

$$y = y_0 \cdot e^{-A(t)} + e^{-A(t)} \int_s^t e^{A(u)} b(u)\,du$$

where

$$A(t) = \int_s^t a(u)\,du\;.$$

The function $e^{-A(t)}$ is the solution of the homogeneous equation taking the value 1 at s. For any other point t $e^{A(t)}e^{-A(u)}$ will be the solution of the homogeneous which takes the value 1 at u. In other words

$$y_{[1],s} = e^{A(t)-A(s)}$$

and we can write the formula as

$$\begin{aligned} y &= y_0 \cdot y_{[1],s} + \int_s^t e^{-A(t)+A(u)} b(u)\,du \\ &= y_0 \cdot y_{[1],s} + \int_s^t y_{\#,u}(t)b(u)\,du \end{aligned}$$

if we set

$$y_{\#,s} = y_{[1],s}$$

which is very close to the formula for second order equations.

This illustrates what happens for a linear differential equation of arbitrary order n. The fundamental solution $y_{\#,s}$ is defined to be the solution $y(t)$ such that

$$y(s) = 0,\ \ldots,\ y^{(n-2)}(s) = 0,\ y^{(n-1)}(s) = 1$$

In other words, the important condition is on the highest possible derivative that can be specified in an initial condition, and varies with the order of the differential equation.

The essential difference between first and higher order equations is that for first order equations we can always find solutions of the associated homogeneous equation, whereas for second order this may not be possible.

Forgetting the initial conditions

If we are only interested in the general solution, it is

$$c_1 z_1 + c_2 z_2 + \int^t y_{[0,1],u}(t)c(u)\,du$$

We can get by with the indefinite integral because for any fixed t_0 and t_1 the integral

$$\int_{t_0}^{t_1} y_{\#,u}(t)c(u)\,du$$

is a solution to the homogeneous equation. This is because

$$y_{\#,s}(t) = c_1(s)z_1(t) + c_2(s)z_2(t)$$

and the integral is

$$z_1(t)\int_{t_0}^{x_1} c_1(u)c(u)\,du + z_2(t)\int_{t_0}^{x_1} c_2(u)c(u)\,du$$

which is after all a linear combination of solutions.