Mathematics 255—February 6, 1996

Midterm examination solutions

(•) *Find the general solution of the differential equation*

$$y'(t) = 2y(t) + 1 + t + \sin t$$

and then the solution such that $y(0) = 0$.

(a) We apply the formula

$$y = Ce^{2t} + e^{2t} \int^t e^{-2s}(1 + s + \sin s)\, ds$$

Since integration by parts gives us

$$\int xe^{\lambda x}\, dx = \frac{xe^{\lambda x}}{\lambda} - \frac{e^{\lambda x}}{\lambda^2}$$

We have

$$\begin{aligned}
\int^t e^{-2s}\, ds &= \frac{e^{-2t}}{-2} \\
\int^t se^{-2s}\, ds &= \frac{te^{-2t}}{-2} - \frac{e^{-2t}}{4} \\
\int^t e^{-2s} \sin s\, ds &= \text{ imaginary part of } \int^t e^{(-2+i)s}\, ds \\
\int^t e^{(-2+i)s}\, ds &= \frac{e^{(-2+i)t}}{(-2+i)} \\
&= e^{-2t}\, \frac{(-2-i)(\cos t + i \sin t)}{5} \\
\int^t e^{-2s} \sin s\, ds &= \frac{e^{-2t}(-2\sin t - \cos t)}{5}
\end{aligned}$$

so

$$y = Ce^{2t} - \frac{1}{2} + \left(-\frac{t}{2} - \frac{1}{4}\right) + \frac{-2\sin t - \cos t}{5}$$

(b) We have

$$0 = C - \frac{3}{4} - \frac{1}{5}, \quad C = \frac{19}{20}$$

(•) *A tank is being filled with water at a rate of* 1 *litre per second, while it is draining out at the bottom, according to Torricelli's Law, at a rate* $\alpha\sqrt{2gh}$, *where* α, *the effective area of the exit hole, is* 2 cm^2. *If the tank has a cross section area of* h^2 *at height* h, *find the equilibrium height. (Gravitational acceleration* g *is* 9.8 *metres/sec*^{2}, *and there are* 1000 cm^3 *in a litre.)*

At equlibrium flow in is the same as flow out:

$$1000 - 2\sqrt{1960h}, \quad h = \left(\frac{500}{\sqrt{1960}}\right)^2 = 127.55\,\text{cm}$$

(•) *Use one step of the Improved Euler's method to find an approximation to* $y(0.1)$ *if* y *is a solution of*

$$y' + xy = \cos x, \quad y(0) = 1$$

Here $F(x,y) = -xy + \cos x$ so the calculation goes

$$\begin{aligned} s_* &= F(0,1) \\ &= 1 \\ y_* &= y_0 + (0.1)s_0 \\ &= 1.1 \\ s_{**} &= F(0.1, y_*) \\ &= -(0.1)(1.1) + \cos(0.1) \\ &= 0.8850 \\ y_1 &= y_0 + (0.05)(1 + 0.8850) \\ &= 1.0943 \end{aligned}$$

(•) *I am using Improved Euler's to estimate $y(1)$, where y is the solution of the same differential equation referred to in the previous question. If I do 10 steps I get an approximate value for $y(1)$ of of 1.19880298, whereas if I do 20 steps I get an approximate value of 1.20031824.*

(a) What is the approximate error in the best value?

(b) What is the best approximation you can get to $y(1)$ using this data?

(c) How many steps would you have to take to get an answer correct to 10 decimals?

(a)

$$\textit{error} = \frac{1.20031824 - 1.19880298}{3} = \frac{0.001515260}{3} = 0.000505087$$

(b)

$$1.20031824 + 0.000505087 = 1.200823327$$

(c) The error for steps of size Δ is $c\,\Delta^2$. For $\Delta = 0.05$ it is 0.000505087, so we solve

$$c = \frac{0.000505087}{(0.05)^2} = 0.20203$$

and then

$$0.20203\Delta^2 = 10^{-10}, \quad \Delta = \frac{10^{-5}}{\sqrt{0.20203}} = 0.000022248, \quad N = 44948$$