Mathematics 255—February 12, 1996

Simply periodic functions

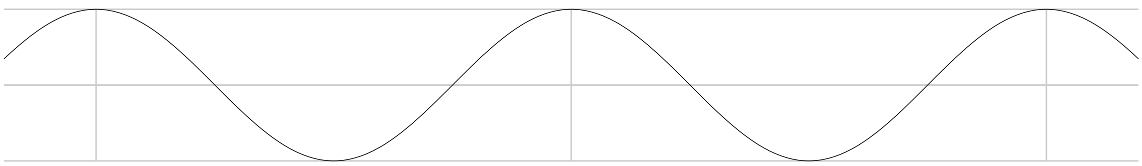
A **simple periodic function** is one of the form $A\cos\omega t + B\sin\omega t$. In these notes we shall characterize such functions in a uniform manner, and see in particular how to draw their graphs.

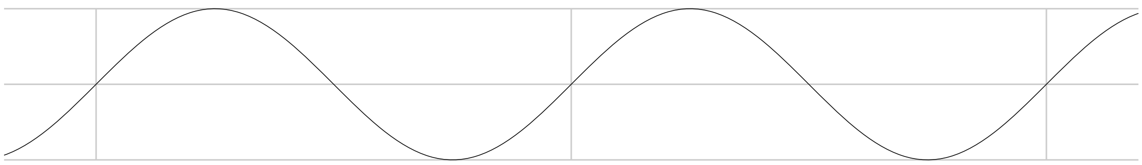
Basics

The simplest functions of this sort are

$$y = \cos t$$
$$y = \sin t$$

They both have period 2π, and their graphs look like this:





These two graphs are in fact essentially the same — they differ only in that the graph of $\sin t$ is obtained from that of $\cos t$ by shifting it to the right by an interval of $\pi/2$, which is one quarter of a period. Now if we are given a function $f(t)$ and shift its graph to the right by an interval of h, we obtain the graph of $f(t-h)$ (for example, at $t = h$ the shifted graph will have the same height as that of f at 0). Therefore the shift in the graphs of $\cos t$ and $\sin t$ is a consequence of the trigonometrical identity

$$\sin t = \cos(t - \pi/2)\ .$$

Of course it is also possible to write

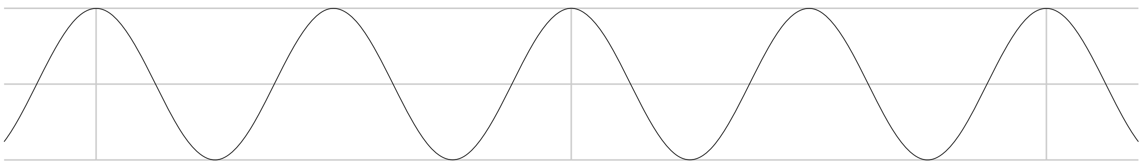
$$\cos t = \sin(t + \pi/2)$$

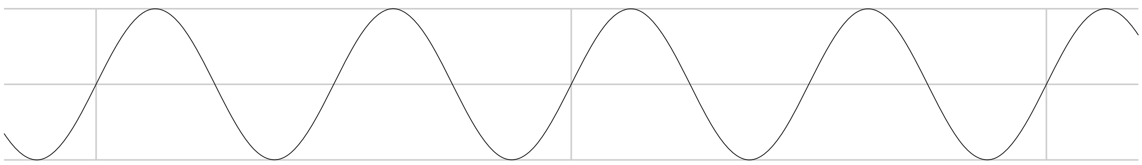
However, it is conventional to use the function $\cos t$ as the basic periodic function, and express all others in terms of it. Thus we shall say that $\cos t$ has a **phase shift** of 0 and that $\sin t$ has a phase shift of $\pi/2$.

Now consider the functions

$$y = \cos 2t$$
$$y = \sin 2t$$

The graphs of these functions are obtained from those of $\cos t$ and $\sin t$ by compressing them along the x-axis by a factor of 2. The **period** of the new functions is equal to π, or half that of the original ones, their **frequency** is twice that of the originals.





In general, the functions

$$\cos \omega t, \quad \sin \omega t$$

will have **period** $T = 2\pi/\omega$ since, for example

$$\cos \omega(t+T) = \cos \omega(t + 2\pi/\omega) = \cos(\omega t + 2\pi) = \cos \omega t$$

In one unit of time it will cover $1/T$ cycles, so that it is also said to have **true frequency**

$$1/T = \omega/2\pi$$

but it is awkward to incorporate the factor of 2π regularly, and it is usually more convenient to refer to the **radian frequency** ω, since in one period the variable will pass through 2π radians.

It is still true that the graph for one of the functions $\cos 2t$ and $\sin 2t$ is obtained from the other by a shift. More precisely, that of $\sin 2t$ is obtained from that of $\cos 2t$ by a shift of $\pi/4$, or half what the previous shift was, since

$$\sin 2t = \cos 2(t - \pi/4)$$

But we can also say that the shift still amounts to a **half cycle**, and rewrite the equation as

$$\sin 2t = \cos(2t - \pi/2)$$

It turns out that it is not the absolute value of the shift which is important, but rather the *amount of shift relative to the length of a cycle*. This is called the **shift in phase**, since in one cycle the phase of a periodic function varies from 0 to 2π. The **phase shift** of the function

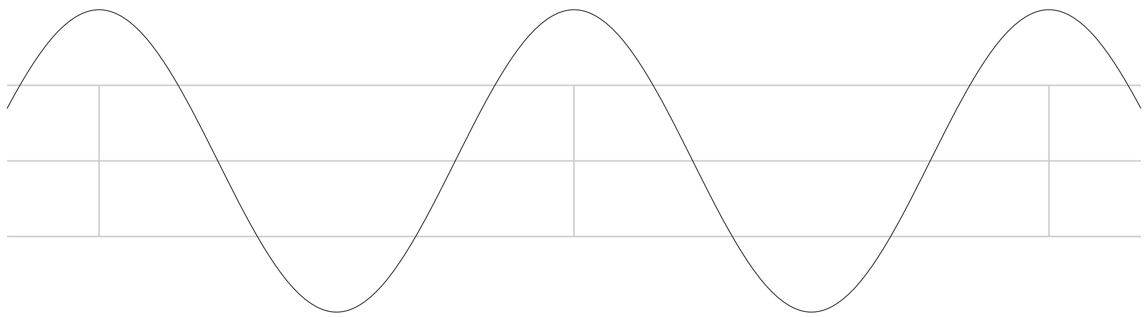
$$\cos(\omega t - \theta) = \cos \omega(t - \theta/\omega)$$

is therefore defined to be θ. The true, or absolute, time shift θ/ω plays only a small role in dealing with periodic functions.

Now consider the function

$$2 \cos t$$

Its relation to $\cos t$ is very simple, since it just oscillates with a greater **amplitude**.



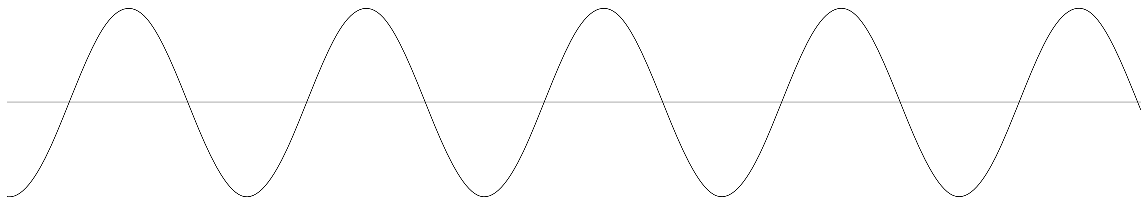
To summarize: a **simple periodic function** is one of the form

$$A\cos(\omega t - \theta)$$

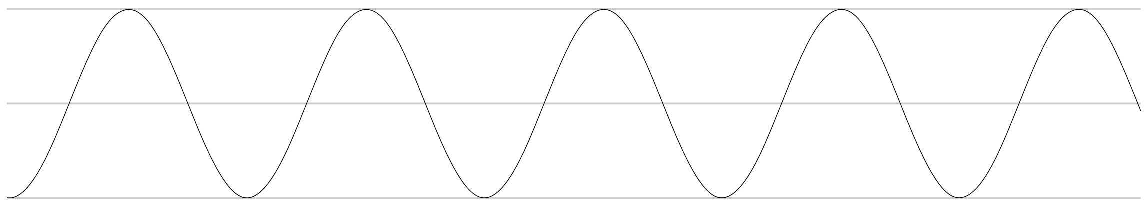
It is said to have **amplitude** A, **radian frequency** ω, **phase shift** θ.

Such a function can be graphed in a very simple but perhaps slightly unorthodox sequence of steps. Consider the curve $y = 1.25\cos(2x - \pi/4)$, for example.

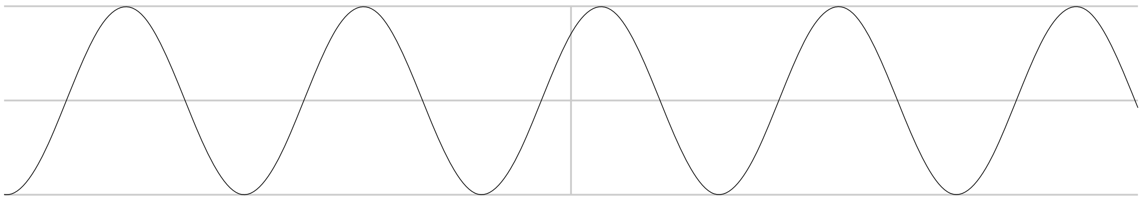
(1) Draw the x-axis and then the graph of $y = \cos t$. This is the unorthodox part of this construction — we draw the graph first and then assemble a framework around it in order to interpret it.



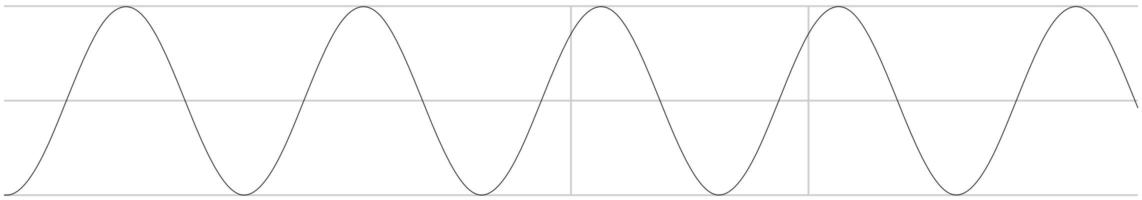
(2) Add the horizontal lines $y = \pm A$ to establish the y-scale.



(3) Add the y-axis to match the phase shift, establishing the origin of the x-axis. Here the shift is 1/8 of a cycle.



(4) Finally establish the x-scale by laying out one period.



The final result we need about simple periodic functions is a generalization of the relationship between $\cos x$ and $\sin x$.

Theorem. *Any linear combination*

$$A \cos \omega x + B \sin \omega x$$

is a simple periodic function.

In other words, it can therefore be expressed in an essentially unique manner

$$A \cos \omega x + B \sin \omega x = C \cos \omega (x - \theta)$$

We must find C and θ. If we write out

$$\cos(\omega x - \theta) = \cos \omega x \cos \theta + \sin \omega x \sin \theta$$

we see that we must have

$$\begin{aligned} C \cos \theta &= A \\ C \sin \theta &= B \end{aligned}$$

If we square these equations and add then since $\cos^2 + \sin^2 = 1$ we get

$$C^2 = A^2 + B^2, \quad C = \sqrt{A^2 + B^2}$$

and once we calculate C we can calculate θ according to the equations

$$\begin{aligned} \cos \theta &= A/C \\ \sin \theta &= B/C \end{aligned}$$

Example. The amplitude of

$$\cos t + \sin t$$

is $C = \sqrt{2}$ and its phase shift is $\pi/4$.

Relations with complex numbers

A complex number z can be written as

$$z = a + ib$$

but also as

$$re^{i\theta}$$

which means that it has polar coordinates r and θ. Sometimes r is called the amplitude of z and θ its phase. The two expressions are related according to the formula

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \cos\theta &= a/r \\ \sin\theta &= b/r \end{aligned}$$

The reason these formulas look so much like the previous ones is this: the function $a\cos(\omega t - \theta)$ may be thought of as the real part of the complex exponential

$$re^{i(\omega t-\theta)} = re^{i\omega t}e^{-i\theta)} = r(\cos\omega t + i\sin\omega t)(\cos\theta - i\sin\theta)$$

whose real part is

$$r\cos\omega t\cos\theta + r\sin\omega t\sin\theta$$

But we can also think of

$$a\cos\omega t + b\sin\omega t$$

as the real part of

$$e^{i\omega t}(a - ib)$$

so we can write

$$a + ib = re^{i\theta}, \quad a - ib = re^{-i\theta}$$

and

$$a\cos\omega t + b\sin\omega t$$

as the real part of

$$re^{i(\omega t-\theta)}$$

We shall see later that this way of using complex numbers can simplify the solution of some differential equations.