

## Mathematics 256 — a course in differential equations for engineering students

### Chapter 1. How things cool off

One physical system in which many important phenomena occur is that where an initial uneven temperature distribution causes heat to flow. Of course as heat flows the temperature distribution changes, which in turn modifies the heat flow. That is to say: Heat flows from hot places to cold ones, and as this happens, the temperature of cold places rises and the temperature of hot places decreases. There are two rough mathematical rules governing the relationship between heat flow and temperature change:

- Heat flow is proportional to spatial rates of change in temperature distribution.
- The time rate of temperature change at any point is proportional to the rate of heat flow into that point.

#### 1. Newton's law of cooling

The general situation can be really complicated, and we shall look at it in some detail later on. But here we shall look at a simple situation, that where a small object is cooling off or heating up in a large environment of fixed temperature. Here the object is cooling off because heat is flowing into or out of it from the environment, and the heat flow takes place because of a temperature difference. There are many ways in which this case is over simplified. One is that any realistic object in the course of changing temperature will have different temperatures at different places, but this can be ignored by assuming the object small enough.

The rate at which a warm object in a cold room cools off depends on how hot it is. Not only does a hot object cool off, but the hotter it is the more rapidly it cools. A somewhat more precise attempt to describe this behaviour is **Newton's law of cooling** which asserts that *the rate at which an isolated object changes temperature is proportional to the difference between its temperature and the surrounding temperature*. If  $\theta(t)$  is the temperature of the object at time  $t$  and  $\theta_{\text{env}}$  is the temperature of its environment, Newton's law can be written

$$\theta' = -k(\theta - \theta_{\text{env}})$$

with  $k$  a positive constant. In other words, illustrating one of the basic principles of this course:

- The natural mathematical expression of Newton's law of cooling is a differential equation of first order.

In general  $\theta_{\text{env}}$  will depend on time. But for the moment we shall just look at the simplest case when it is held constant. Intuitively, we expect the object's temperature simply to approach  $\theta_{\text{env}}$  as time goes on. Suppose, for example, that  $\theta_{\text{env}} = 0$ . Then Newton's Law becomes

$$\frac{d\theta}{dt} = \theta' = -k\theta .$$

This is an equation you have seen in your calculus course. We can solve it by dividing both sides by  $\theta$  and then rearranging to get

$$\frac{d\theta}{\theta} = -k dt .$$

If we integrate both sides we get

$$\log \theta = -kt + c$$

for some constant of integration  $c$  or

$$\theta = C e^{-kt}$$

for some other constant  $C = e^c$ . If we set  $t = 0$  we see that  $C$  must be the initial temperature  $y(0)$  of the object. In other words, in this environment the temperature will decrease exponentially towards 0.

Whether  $\theta_{\text{env}} = 0$  or not cannot be significant, since the level at which zero temperature is set is rather arbitrary. To solve the general equation we rewrite it as

$$(\theta - \theta_{\text{env}})' = -k(\theta - \theta_{\text{env}})$$

which amounts to a shift in origin to the environmental temperature. Then we get the solution

$$\theta - \theta_{\text{env}} = C e^{-kt}, \quad \theta = \theta_{\text{env}} + C e^{-kt}$$

for some constant  $C$ . The constant  $C$  is determined by the conditions in which things started out. Suppose the initial temperature at  $t = 0$  to be  $\theta_0$ . If we set  $t = 0$  then the formula gives us  $\theta_0 = \theta_{\text{env}} + C$ , so that

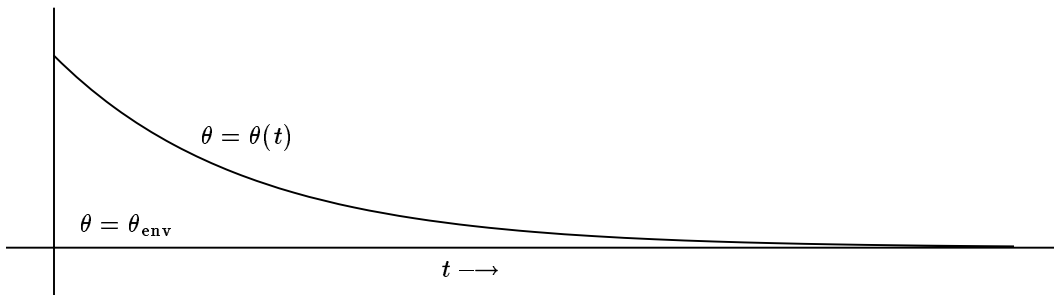
$$C = \theta_0 - \theta_{\text{env}},$$

the initial temperature difference. All in all, we get the solution

$$\theta = \theta_{\text{env}} + (\theta_0 - \theta_{\text{env}})e^{-kt}.$$

This says that *the difference between the temperature of the object and the surrounding object decreases at an exponential rate.*

If the initial temperature is the same as the environmental temperature, it doesn't change. In other words, the constant temperature function  $\theta(t) = \theta_{\text{env}}$  is itself a solution of the differential equation. It is called an **equilibrium** solution of the equation.



This sort of approach to equilibrium in physical systems is very common. It is characterized here by the property that *in any fixed amount of time the difference between temperature and equilibrium is multiplied by a constant factor.* To be precise, in a time interval  $\Delta t$  the temperature difference scales by  $e^{-k \Delta t}$ .

- The amount of time it takes for the difference to scale by the factor  $e^{-1}$  is called the **relaxation time**  $\tau$  of the system.

If we solve  $e^{-k \Delta t} = e^{-1}$  we see that

$$\Delta t = \tau = \frac{1}{k}$$

and we can rewrite the formula for  $\theta$  as

$$\theta(t) = (\theta_0 - \theta_{\text{env}})e^{-t/\tau} + \theta_{\text{env}}.$$

**Exercise 1.1.** A cup of coffee initially at  $90^\circ$  sitting in a room at  $20^\circ$  takes 5 minutes to cool to  $70^\circ$ . (a) What is the coffee temperature after 10 minutes? (b) What is the relaxation time? (c) How long does it take to cool to  $40^\circ$ ? (d) Write down a formula for the temperature at time  $t$  (given in minutes).

## 2. The mathematics abstracted

What we have said is not tied necessarily to the physics of a cooling object. There are many circumstances in which we want to solve a first order differential equation

$$y' = ay + b$$

where  $a$  and  $b$  are constants. We can rewrite this equation as

$$\frac{dy}{ay + b} = dt$$

and integrate both sides, substituting  $u = ay + b$ , to get

$$\begin{aligned} \frac{du}{au} &= dt \\ \log(ay + b) &= at + c \\ ay + b &= e^c e^{at} \\ y &= C e^{at} - \frac{b}{a} \end{aligned}$$

for some suitable constant  $C$ . If we set  $t = 0$ , we get

$$y_0 = C - \frac{b}{a}$$

so  $C = y_0 + b/a$ . Explicitly then the solution is

$$y = (y_0 + b/a)e^{at} - b/a = y_0 + \left(\frac{b}{a}\right)(e^{at} - 1)$$

if  $y = y_0$  at  $t = 0$ .

We shall see in a later section that if we look at an object cooling in an environment whose temperature is changing with time, we arrive at a differential equation of the form

$$y' = ay + b(t)$$

where now  $b(t)$  might depend on time. We can find the general solution of this equation by a trick. Rewrite it as

$$y' - ay = b(t) .$$

Multiply both sides by  $e^{-at}$  to get

$$y' e^{-at} - ay e^{-at} = e^{-at} b(t) .$$

Notice that the left hand side is the derivative of  $ye^{-at}$  according to the product rule for derivatives. Therefore we have

$$\frac{d ye^{-at}}{dt} = e^{-at} b(t)$$

which means that  $e^{-at}y$  is the indefinite integral of  $e^{-at}b(t)$ . In a slightly pedantic formulation

$$ye^{-at} = \int^t e^{-as} b(s) dt, \quad y = e^{at} \int^t e^{-as} b(s) ds .$$

We need the variable  $s$ , which is called a *dummy variable* or *variable of integration* in calculus, because it *must* be distinguished from  $t$ . The integral here is indefinite. If we take an integration constant into account we can write this formula a bit more suggestively:

- *The general solution to the differential equation*

$$y' = ay + b(t)$$

is

$$y = Ce^{at} + e^{at} \int e^{-as} b(s) ds .$$

where  $C$  is an arbitrary constant.

The integral here is still indefinite, undetermined up to an additive constant. Any such constant will affect the constant  $C$ , so that we cannot expect  $C$  to have any particular meaning unless we fix that integration constant. We can do this by changing the indefinite integral into a definite one:

$$y = Ce^{at} + e^{at} \int_{t_0}^t e^{-as} b(s) ds$$

where  $t_0$  is some arbitrarily chosen moment of time. Now if we set  $t = t_0$  the integral becomes 0 and we get  $y(t_0) = Ce^{at_0}$  which gives

$$C = e^{-at_0} y(t_0)$$

and we have:

- *The solution to the equation*

$$y' = ay + b(t)$$

such that  $y(t_0) = y_0$  is given by the formula

$$y = y_0 e^{a(t-t_0)} + e^{at} \int_{t_0}^t e^{-as} b(s) ds .$$

**Exercise 2.1.** A radioactive substance emits radiation and changes to a new substance. The amount of radioactivity is proportional to the quantity of the original substance remaining. The half-life of a radioactive substance is the amount of time it takes for half the substance to decay radioactively. If the half life of radium is 1760 years, how much radium is left from an initial gram of radium after 100 years?

**Exercise 2.2.** What is the exact relationship between half-life and relaxation time?

**Exercise 2.3.** Solve the differential equation & initial conditions

$$y' + y = t, \quad y(0) = 0 .$$

### 3. When the right hand side is an exponential function

In many practical cases,  $b(t) = e^{ct}$ . In this case the solution of  $y' = ay + e^{ct}$  has a particularly simple form. We apply the integral formula to get

$$\begin{aligned} y &= Ce^{at} + e^{at} \int e^{-as} e^{cs} ds \\ &= Ce^{at} + e^{at} \int e^{(c-a)s} ds \\ &= Ce^{at} + e^{at} \frac{e^{(c-a)t}}{c-a} \quad (c \neq a) \\ &= Ce^{at} + \frac{e^{ct}}{c-a} \\ &= Ce^{at} + te^{at} \quad (c = a) \end{aligned}$$

**Exercise 3.1.** Suppose we have two radioactive substances  $A$  and  $B$ . The substance  $A$  decays into  $B$  with a relaxation time of  $\tau_A$ , and  $B$  decays into the stable substance  $C$  with relaxation time  $\tau_B$ . Let  $y_A$ ,  $y_B$ ,  $y_C$  be the amounts of each substance (depending on time). Write down three first order differential equations describing the relationship between the three quantities.

**Exercise 3.2.** (Continuing) Assuming that we start with an amount  $y_0$  of substance  $A$  at  $t = 0$ , find an expression for  $y_A$  and  $y_B$  at time  $t$ .

**Exercise 3.3.** Suppose are looking at a cup of coffee initially at  $100^\circ$  where the relaxation time is 20 minutes, in an environment whose temperature is fluctuating at a simple frequency of period 10 minutes according to the formula

$$\theta_{\text{env}}(t) = 5 \cos 2\pi t/10 .$$

What is the temperature of the coffee after 20 minutes?

This exercise raises a number of questions which might require some review. We shall take up this question after a few digressions.

#### 4. Separable first order equations

We have seen above one technique for solving first order differential equations which works rarely, but which occurs often enough to call attention to it. We solved the equation

$$\frac{dy}{dt} = y' = ay + b$$

by rewriting it as

$$\frac{dy}{ay + b} = dt$$

and then integrating both sides. When we can rewrite a first order equation

$$y' = f(t, y)$$

so all occurrences of  $y$  are on one side and all occurrences of  $t$  are on the other, the equation is said to be **separable**. We can then integrate to find an implicit equation for  $y$  in terms of  $t$ . Usually this equation does not tell us much, even when it can be found.

**Exercise 4.1.** An object at radius  $r$  falling towards Earth satisfies the first order differential equation

$$r' = -\sqrt{\frac{2k}{r} + E}$$

where  $E$  is the (constant) energy of the object. Solve this by separation of  $r$  and  $t$ , integrate what you get (by substitution  $u = 2k/r + E$ ), and find an implicit equation for  $r$  in terms of  $t$ .

**Exercise 4.2.** The rate of evaporation of a spherical raindrop is proportional to its surface area. If it takes 5 minutes to evaporate from 3 mm. radius to 2 mm. radius, how much longer does it take to disappear?

**Exercise 4.3.** Find a formula for general solution of

$$y' = a(t)y .$$

**Exercise 4.4.** By mimicking the derivation for the solution of  $y' = ay + b(t)$ , find a formula for the general solution of

$$y' = a(t)y + b(t) .$$