Chapter 12. Review of eigenvalues and eigenvectors

It will be as short as possible.

1. General review

If A is a matrix, an **eigenvector** of A is a vector ξ which is (a) not the zero vector and (b) taken into a multiple of itself through left multiplication by A:

$$A\xi = \lambda\xi$$

for some constant λ. This equation can be rewritten as

$$(A - \lambda I)\xi = 0$$

where I is the identity matrix of the same size as A. Say A has size $n \times n$. Then this equation amounts to n equations in the n unknown coordinates of ξ. In general, n linear equations in n unknowns will have exactly one solution. In this case, the system certainly has the zero vector as a possible candidate for ξ, but the definition of eigenvector doesn't allow that candidate. In fact, λ will be an eigenvalue for A precisely when the system of equations with coefficient matrix $A - \lambda I$ has more than one solution. This happens when $A - \lambda I$ is not an invertible matrix, or when

$$\det(A - \lambda I) = 0$$

which is to say that the matrix $A - \lambda I$ is **singular**. The determinant $\det(A - \lambda I)$ is a polynomial in the variable λ of degree n. It is called the **characteristic polynomial** of A.

- *The eigenvalues of A are the roots of the characteristic polynomial of A.*
- *If λ is an eigenvalue of A then the eigenvectors for this eigenvalue are the solutions of the singular system of equations*

  $$(A - \lambda I)\xi = 0$$

  other than the zero vector.

2. Two by two matrices

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

and its characteristic polynomial is

$$(a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + (ad-bc) = \lambda^2 - \lambda\,\mathrm{trace}(A) + \det(A)$$

where the **trace** of A is the sum $a+d$ of its diagonal entries.

Example.

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

Its characteristic polynomial is

$$\lambda^2 - 4\lambda + 3$$

and its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$. For $\lambda = 1$ we have

$$A - \lambda I = A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and the system of eigenvector equations $(A - \lambda I)\xi = 0$ becomes

$$\begin{aligned} x + y &= 0 \\ x + y &= 0 \ . \end{aligned}$$

These two equations are the same, or in other words the system is redundant. This is essentially what has to happen, because for an eigenvalue the system must be singular. At any rate, the eigenvectors for $\lambda = 1$ are the points on the line $y = -x$ (except for $(0, 0)$). All are multiples of the single vector

$$\xi_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ .$$

For $\lambda = 3$ the eigenvector equations are

$$\begin{aligned} -x + y &= 0 \\ x - y &= 0 \end{aligned}$$

which are again redundant. The eigenvectors for $\lambda = 3$ are multiples of

$$\xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ .$$

Notice that the matrix A is symmetric. The orthogonality of the eigenvectors is consistent with a claim in another section.

Example.

Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \ .$$

Its characteristic polynomial is

$$\lambda^2 + 2\lambda + 2 = 0$$

with roots

$$\lambda = 1 \pm i \ .$$

The eigenvector equations for $1 + i$ is

$$\begin{aligned} -ix - y &= 0 \\ x - iy &= 0 \ . \end{aligned}$$

These are redundant since the second is i times the first. We have

$$\xi_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \ .$$

In this case, we do not have to deal separately with the other eigenvalue, since it is the **complex conjugate** of the first. We take as eigenvector the conjugate of the first eigenvector

$$\xi_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} .$$

If A is an arbitrary 2×2 matrix then there are essentially three ways in which the eigenvalues and eigenvectors of A can behave. • The eigenvalues are real and we can find two linearly independent eigenvectors. • The eigenvalues are conjugate complex numbers, and we can find conjugate complex eigenvectors as well. • There is an exceptional case also, in which A has just one eigenvalue—real—but also only a single line of eigenvectors. We shall see later to what extent we need to understand this case. Here I only mention that

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is an example.

Exercise 2.1. *Find eigenvalues and eigenvectors for*

$$\begin{bmatrix} -5 & 4 \\ 4 & -6 \end{bmatrix} .$$

Exercise 2.2. *Find eigenvalues and eigenvectors of*

$$\begin{bmatrix} -a & a \\ a & -a \end{bmatrix} .$$

Exercise 2.3. *Find eigenvaluesa nd eigenvectors of*

$$\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix} .$$

Exercise 2.4. *Find eigenvalues and eigenvectors of*

$$\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} .$$

3. Symmetric matrices

Eigenvalues of arbitrary matrices can be complex numbers, and a number of exceptional phenomena can occur in relating eigenvalues to eigenvectors. There is one general circumstance, however, in which things are relatively simple. Recall that a matrix is **symmetric** if it is equal to its own transpose. If A is a symmetric matrix then its eigenvalues are always all real, and we can find a set of eigenvectors which are all orthogonal, and as many of them as dimensions to the vectors involved. This happens often with physical systems, as it did with the system of weights on springs, where the matrix K was symmetric.

These facts about symmetric matrices are so important that I shall review them here. We shall do this for arbitrary $n \times n$ symmetric matrices. The starting point is to understand how dot products and matrix multiplication relate to each other.

If

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$$

are two n-dimensional vectors then their dot product is the sum

$$u \bullet v = u_1 v_1 + u_2 v_2 + \cdots u_n v_n \ .$$

- *The dot product of two column vectors u and v is the same as the matrix product*

$$^t\!u\, v = [\, u_1 \quad u_2 \quad \dots \quad u_n \,] \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \ .$$

- *If M is any $n \times n$ matrix then*

$$Mu \bullet v = u \bullet {}^t\!M v \ .$$

The first claim is a simple observation. The second follows from it, since we have

$$Mu \bullet v = {}^t(Mu)\, v = {}^t\!u\, {}^t\!M\, v = u \bullet {}^t\!M v \ .$$

The main consequence for us is:

- *If M is any $n \times n$ symmetric matrix then*

$$Mu \bullet v = u \bullet Mv \ .$$

- *The eigenvalues of any real symmetric matrix are all real. If $\lambda_1 \neq \lambda_2$ are distinct eigenvalues for eigenvectors ξ_1 and ξ_2 then $\xi_1 \bullet \xi_2 = 0$.*

We shall need a simple fact about complex numbers. If $z = x + iy$ then its conjugate is $\overline{z} = x - iy$ and

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 \geq 0 \ .$$

This cannot be 0 unless $z = 0$. If v is a complex vector then the dot product of v and $\overline{v}$ is

$$v \bullet \overline{v} = v_1\overline{v}_1 + v_2\overline{v}_2 + \cdots + v_n\overline{v}_n \ .$$

Each term in this sum is non-negative and therefore the sum can be 0 only if all $v_i = 0$.

If M is a (real) symmetric matrix and $Mv = \lambda v$ then we also have $M\overline{v} = \overline{\lambda}\overline{v}$ and

$$\begin{aligned} Mv \bullet \overline{v} &= \lambda v \bullet \overline{v} \\ &= \lambda\, (v \bullet \overline{v}) \\ &= v \bullet M\overline{v} \\ &= v \bullet \overline{\lambda}\overline{v} \\ &= \overline{\lambda}\, (v \bullet \overline{v}) \ . \end{aligned}$$

But then

$$(\lambda - \overline{\lambda})\, v \bullet \overline{v} = 0$$

If v is an eigenvector then by definition $v \neq 0$ so $v \bullet \overline{v} \neq 0$. Therefore $\lambda = \overline{\lambda}$ which means that λ is real.

If $Mv_1 = \lambda_1 v_1$ and $Mv_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$ then

$$\begin{aligned} Mv_1 \bullet v_2 &= \lambda_1\ (v_1 \bullet v_2) \\ &= v_1 \bullet Mv_2 \\ &= \lambda_2\ (v_1 \bullet v_2) \end{aligned}$$

and since the λ_i are different, we must have $v_1 \bullet v_2 = 0$.

If all the eigenvalues of a symmetric $n \times n$ matrix M are distinct, then this result implies that we can find n eigenvectors v_i, all perpendicular to each other, none of course 0. But even without the assumption of distinctness, a little care will give us the same result.

- *If M is an $n \times n$ symmetric matrix then we can find a set of n eigenvectors v_i for M with $v_i \bullet v_j = 0$ for $i \neq j$.*

If v is any n dimensional vector, we can then write

$$v = \sum c_i c_i, \qquad c_i = \frac{v \bullet v_i}{v_i \bullet v_i}\ .$$