Chapter 7. Homogeneous equations with constant coefficients

It has already been remarked that we can write down a formula for the general solution of any linear second differential equation

$$y'' + a(t)y' + b(t) = f(t)$$

but that it would not be so explicit as the formula for first order linear equations. In fact, it is a formula that is almost useless unless we make some special assumption about the equation. The reason for this is that the formula involves the general solution to the homogeneous equation—the associated equation where $f(t) = 0$. For this there is bad news and good news:

- *There is no general formula for solving second order homogeneous linear differential equations.*
- *In case the homogeneous linear equation has constant coefficients, however, there is a way to find all of its solutions.*

In other words, what the formula does is solve the inhomogeneous case in terms of the homogeneous case, but we are usually stopped right there. But–there is one large category of equations which can be solved, however—those linear homogeneous equations where the coefficients $a(t)$ and $b(t)$ are constants not depending on t. Even in this relatively simple case there is much to be said, and we shall spend a fair amount of time in this chapter on this topic.

1. Linear equations with constant coefficients

Let's now consider such an equation, which we shall write in a form slightly different from what we saw earlier:

$$y'' + ay' + by = 0 \ .$$

To get some feel for what will happen, let's look at an explicit example

$$y'' - y = 0 \ .$$

This can be written in the old form

$$y'' = y$$

so that solutions of this differential equation are those functions $y(t)$ whose double derivatives are equal to themselves. There is certainly one function $y(t) = e^t$ which comes to mind quickly. Another and similar function is $y(t) = e^{-t}$, since for this function $y' = -y$, $y'' = -(-y) = y$. Now the solutions to any linear homogeneous equation satisfy the linearity principle, so we can obtain lots more solutions by taking linear combinations

$$ae^t + be^{-t}$$

of the ones we already have. The converse turns out also to be true.

- *Any solution of the differential equation*

  $$y'' - y = 0$$

  can be expressed as a linear combination

  $$ae^t + be^{-t} \ .$$

This is another way of saying that we can assign coordinates, here the constants a and b, to every solution of the equation.

The proof involves a calculation we shall repeat often. The point of the argument is to reduce the claim to the fact we already know: *Any second order equation is determined by the value of y and y' at any fixed point t_0.*

The first step is to construct first the fundamental solutions associated to $t = 0$ from the solutions e^t, e^{-t}. The fundamental solution y_0 for example satisfies

$$\begin{aligned} y_0(0) &= 1 \\ y_0'(0) &= 0 \ . \end{aligned}$$

We speculate that y_0 is a linear combination of e^t and e^{-t}. Set

$$y_0 = ae^t + be^{-t}, \quad y_0' = ae^t - be^{-t} \ .$$

Then

$$y_0(0) = a + b, \quad y_0'(0) = a - b \ .$$

We must see if we can choose a and b so that

$$\begin{aligned} a + b &= 1 \\ a - b &= 0 \ . \end{aligned}$$

This gives us two equations in the two unknowns a and b, and has the solution

$$a = 1/2, \quad b = 1/2 \ .$$

Therefore

$$y_0 = y_{0,0} = \frac{e^t + e^{-t}}{2} \ .$$

Similarly we can calculate that

$$y_1 = y_{1,0} = \frac{e^t - e^{-t}}{2} \ .$$

But now if we are given any solution of the differential equation we can write it as a linear combination of y_0 and y_1, according to reasoning I presented earlier, and then we can write it in turn as a linear combination of e^t and e^{-t}.

Much of what I have said here applies to more general linear differential equations with constant coefficients. The first step is to realize that solving such equations always involves exponential functions of some kind. To see why this should be so, and to understand why exponential functions arise in this context, is important and interesting in its own right.

- *Any homogeneous linear differential equation–of any order whatsoever—with constant coefficients has at least one solution of the form*

  $$y = e^{\lambda t}$$

  for a suitable choice of λ.

In solving the equation, we will therefore simply have to find what λ has to be. It turns out that this reduces to a simple problem in algebra. But before I discuss the algebra involved, I will discuss in the next section why this extraordinary result is true. This discussion will illustrate an important way of thinking about linear differential equations with constant coefficients, and introduces for them an important principle which distinguishes them from linear differential equations with variable coefficients.

Exercise 1.1. *Find exponential solutions of $y'' - 4y = 0$, $y'' + y = 0$. Find also fundamental solutions at 0.*

2. Time invariance and the translation principle

Physical systems which give rise to linear second order equations with constant coefficients are **time invariant**. I have said that a differential equation is often the mathematical formulation of Newton's Law. The coefficients which occur are physical quantities associated to the system, and they will be constant precisely when the physical situation that the differential equation encodes is one which does not vary with time. The *system* is time invariant, but this is not the same as saying that the *solutions of the system* are time-invariant. What it does mean is that if we shift the initial conditions in time, then the solutions will likewise be shifted in time. Mathematically, a shift by time interval h means replacing a function $f(t)$ by $f(t-h)$, and the **principle of time invariance** for constant coefficient equations is this:

- *If $y(t)$ is a solution of a linear homogeneous differential equation with constant coefficients, then for any constant h so is $y(t-h)$.*

Keep in mind: it is not the particular motions of a system which are time invariant, but the underlying physical law which controls its dynamics. This statement about time shifts is the way to formulate this precisely.

A very closely related fact is this:

- *If $y(t)$ is a solution of a linear homogeneous differential equation with constant coefficients, then so is its derivative $y'(t)$.*

The way in which these two principles are related is that

$$y' = \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}$$

so that the derivative of a function is the limit of a linear combination of y and its shifts.

Both of these are straightforward to prove. The second, for example: Suppose that y is a solution of the linear equation

$$y'' + a(t)y' + b(t)y = 0$$

where for the moment I don't assume a and b to be constants. Take the derivative of both sides:

$$y''' + a'(t)y' + a(t)y'' + b'(t)y + b(t)y' = 0 .$$

But now if we assume $a(t)$ and $b(t)$ are constant, then $a'(t) = 0$ and $b'(t) = 0$, and the equation becomes

$$(y')'' + a(y')' + b(y') = 0$$

which means that y' is also a solution.

So we are now in the following circumstances: we have an operator $y \mapsto y'$ on the set of all solutions of the differential equation, and the set of solutions is a two dimensional vector space. The operator is linear, because it clearly satisfies scalability and superposition. This means that if we express it in terms of coordinates, we get a 2×2 matrix. But now one thing we know about 2×2 matrices is that they always have at least one eigenvalue. In our circumstances, this means that there exists at least one solution y of the differential equation which is taken into a multiple of itself by the differentiation operator.

$$y' = \lambda y$$

for some constant λ, the eigenvalue. However, this eigenvalue equation is nothing but a first order linear differential equation! We know that it has as solution the exponential function $e^{\lambda t}$.

- *Linearity together with time invariance imply that we can find at least one solution which is an exponential function.*

Exercise 2.1. *The method we apply to second order equations works for the equation $y''' = y$, too. Find one exponential solution. Find all exponential solutions.*

3. Finding explicit solutions

We now look into what we have to do to find an exponential solution $y = e^{\lambda t}$ to the linear homogeneous differential equation with constant coefficients

$$y'' + ay' + by = 0 .$$

Of course we just set $y = e^{\lambda t}$ and substitute it into the equation. We have

$$y = e^{\lambda t}$$
$$y' = \lambda e^{\lambda t}$$
$$y'' = \lambda^2 e^{\lambda t}$$

and

$$y'' + ay' + by = (\lambda^2 + a\lambda + b)e^{\lambda t} .$$

To obtain a solution of the equation we must set λ equal to a root of the **characteristic equation**

$$\lambda^2 + a\lambda + b = 0 .$$

Example.

$$y'' - y = 0$$

Here the characteristic equation is

$$\lambda^2 = 1$$

and we get the same solutions

$$e^t, \quad e^{-t}$$

we got before.

Example.

$$y'' - 3y' + 2y = 0$$

The characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

with roots $\lambda = 2, 1$ and we get solutions

$$e^{2t}, \quad e^t$$

- *The simplest case is when the characteristic equation has two distinct roots $\lambda_1 \neq \lambda_2$. Then the general solution to the equation is*

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} .$$

Example.

$$y'' + y = 0$$

Characteristic equation

$$\lambda^2 + 1 = 0$$

Here the roots are $\lambda = \pm i$, and we get solutions

$$e^{it}, \quad e^{-it} .$$

What good are these? We recall Euler's formula

$$e^{it} = \cos t + i \sin t$$

and write

$$\begin{aligned} e^{it} &= \cos t + i \sin t \\ e^{-it} &= \cos t - i \sin t \ . \end{aligned}$$

We now recall that linear combinations of solutions are solutions. Adding and then subtracting the expressions above we get solutions

$$\begin{aligned} e^{it} + e^{-it} &= 2 \cos t \\ e^{it} - e^{-it} &= 2i \sin t \end{aligned}$$

Dividing by constants is also allowed, so we also get solutions

$$\begin{aligned} \frac{e^{it} + e^{-it}}{2} &= \cos t \\ \frac{e^{it} - e^{-it}}{2i} &= \sin t \end{aligned}$$

and these are the solutions which are of interest to us.

- *When we have complex roots of the characteristic equation we also have as solutions their real and imaginary parts, which we calculate according to Euler's formula. More explicitly, if $\lambda = a + ib$ is a root of the characteristic equation then the general solution of the equation is*

$$c_1 e^{at} \cos bt + c_2 e^{at} \sin bt \ .$$

Example. For

$$y'' + y' + y = 0$$

the roots are

$$-\frac{1}{2} + \frac{\sqrt{-3}}{2}$$

and the solutions are

$$e^{-t/2} \cos(\sqrt{3}t/2), \quad e^{-t/2} \sin(\sqrt{3}t/2)$$

Example.

$$y'' = 0$$

Here the characteristic equation is

$$\lambda^2 = 0$$

so we get a single root $\lambda = 0$, and a solution $y = e^{0\,t} = 1$. How do we get another solution? It is trivial to see here that it is $y = t$.

- *When the characteristic equation has just a single root λ then the general solution to the equation is*

$$c_1 e^{\lambda t} + c_2 t e^{\lambda t} \ .$$

When there is only a single root, the characteristic polynomial must be of the form

$$(\lambda - c)^2$$

and the differential equation looks like

$$y'' - 2c\, y' + c^2 = 0\ .$$

In this case you can verify explicitly that te^{ct} does satisfy the equation.

Exercise 3.1. *Prove that if*

$$y'' + ay + by = 0$$

has a characteristic equation with only one root λ*, then* $te^{\lambda t}$ *is a solution.*

Exercise 3.2. *Find general solutions of*

$$\begin{array}{ll} (a) & y'' + 5y' + 6y &= 0 \\ (b) & y'' + 4y' + 4y &= 0 \\ (c) & y'' + y' + y &= 0 \\ (d) & y'' - y' - y &= 0 \\ (e) & y'' + 2y' + y &= 0 \\ (f) & y'' + 4y' + 3y &= 0 \\ (g) & y'' + 8y' - 9y &= 0 \\ (h) & y'' - 2y' + y &= 0 \\ (i) & y'' + 2y' + 2y &= 0 \end{array}$$

4. Initial conditions

The procedure described above gives us two different solutions of the differential equation. All other solutions will be linear combinations of these two. Solving for initial conditions amounts to solving two equations for the two unknown coefficients.

Example. To find the solution of

$$y'' - 3y' + 2y = 0$$

such that $y(0) = 0$, $y'(0) = 1$ we set

$$y = Ae^t + Be^{2t}\ .$$

Then $y' = Ae^t + 2Be^{2t}$ and we get

$$\begin{aligned} A + B &= 0 \\ A + 2B &= 1 \end{aligned}$$

which leads to $A = -1$, $B = 1$. The solution we want is $e^{2t} - e^t$.

Exercise 4.1. *Find the solution of* $y'' + 4y' + 3y = 0$ *such that* $y(0) = 1$, $y(1) = 1$.

5. More about fundamental solutions

If

$$y'' + a(t)y' + b(t)y = 0$$

is any second order linear homogeneous differential equation, then the **fundamental solutions** with respect to the point t_0 are the two solutions $y_0(t)$, $y_1(t)$ such that

$$\begin{array}{ll} y_0(t_0) = 1 & y_0'(t_0) = 0 \\ y_1(t_0) = 0 & y_1'(t_0) = 1 \ . \end{array}$$

I recall that since these solutions depend on the parameter t_0, we write them as y_{0,t_0}, y_{1,t_0}.

One reason they are useful is that if you want to solve the equation above with initial conditions

$$y(t_0) = y_\#, \quad y'(t_0) = v_\#$$

then you just set

$$y = y_\# \cdot y_{0,t_0} + v_\# \cdot y_{1,t_0} \ .$$

But usually it is more work to find y_{0,t_0} and y_{1,t_0} than to solve for the initial conditions directly.

Example. Take

$$y'' - y = 0, \quad t = 0 \ .$$

All solutions are combinations of e^t and e^{-t}, so we set

$$y = c_1 e^t + c_2 e^{-t}$$

and solve first the system

$$\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= c_1 - c_2 = 0 \end{aligned}$$

to get

$$y_{0,0} = (1/2)(e^t + e^{-t}) \ .$$

Next we solve

$$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= c_1 - c_2 = 1 \end{aligned}$$

to get

$$y_{1,0} = (1/2)(e^t - e^{-t}) \ .$$

Example. For

$$y'' + y = 0, \quad t = 0$$

we have

$$\begin{aligned} y_{0,0} &= \cos t \\ y_{1,0} &= \sin t \ . \end{aligned}$$

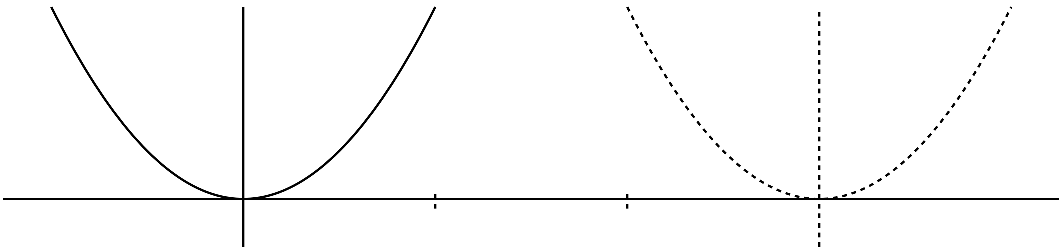
Exercise 5.1. *Find the second fundamental equation at $t = 0$ for each of these equations:*

$$\begin{array}{lll} (a) & y'' + 5y' + 6y & = 0 \\ (b) & y'' + 4y' + 4y & = 0 \\ (c) & y'' + y' + y & = 0 \\ (d) & y'' - y' - y & = 0 \\ (e) & y'' + 2y' + y & = 0 \\ (f) & y'' + 4y' + 3y & = 0 \\ (g) & y'' + 8y' - 9y & = 0 \\ (h) & y'' - 2y' + y & = 0 \\ (i) & y'' + 2y' + 2y & = 0 \end{array}$$

Exercise 5.2. *Graph each of the solutions from the previous exercise.*

6. Fundamental solutions and the translation principle

The graph of the function $y(t)$ when shifted to the right by t_0 is the graph of the function $y(t-t_0)$. This new function of x is called the **translation** of y by t_0. Just below, for example, are the graphs of $y=t^2$ and $y=(t-3)^2$.



I recall the **translation principle** for equations with constant coefficients. If $y(t)$ is a solution of the differential equation

$$y''(t)+a(t)y'(t)+b(t)y(t)=0$$

then also for any t_0

$$y''(t-t_0)+a(t-t_0)y'(t-t_0)+b(t-t_0)y(t-t_0)=0\ .$$

In particular, if $a(t)$ and $b(t)$ are constants we see that *if $y(t)$ is a solution of a differential equation with constant coefficients then so are all translations $y(t-t_0)$*. I also recall that this is related to the principle that *derivatives of solutions are solutions*, since if $y(t)$ is a solution so is $y(t+h)$, and then according to linearity so is $[y(t+h)-y(t)]/h$. But then

$$y'(t)=\lim_{h\to 0}\frac{y(t+h)-y(t)}{h}\ .$$

If $z(t)=y(t-t_0)$ then $z(t_0)=y(0)$ and $z'(t_0)=y'(0)$. Therefore for differential equations with constant coefficients

$$\begin{aligned}y_{0,t_0}&=y_{0,0}(t-t_0)\\ y_{1,t_0}&=y_{1,0}(t-t_0)\ .\end{aligned}$$

In other words

- *to find the fundamental solutions for any point t_0 we have only to find those for $t_0=0$ and translate them by t_0.*