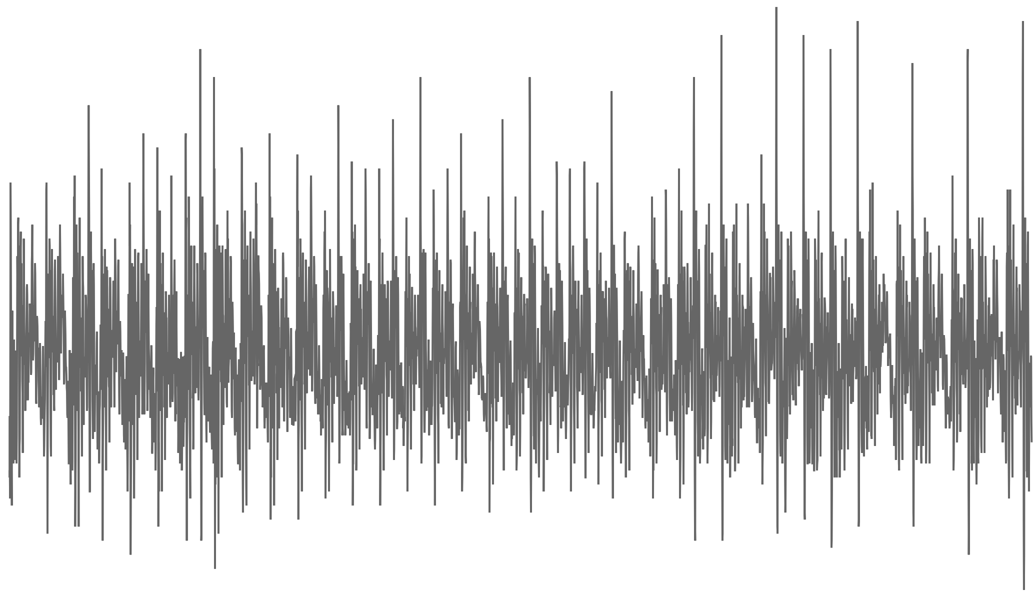
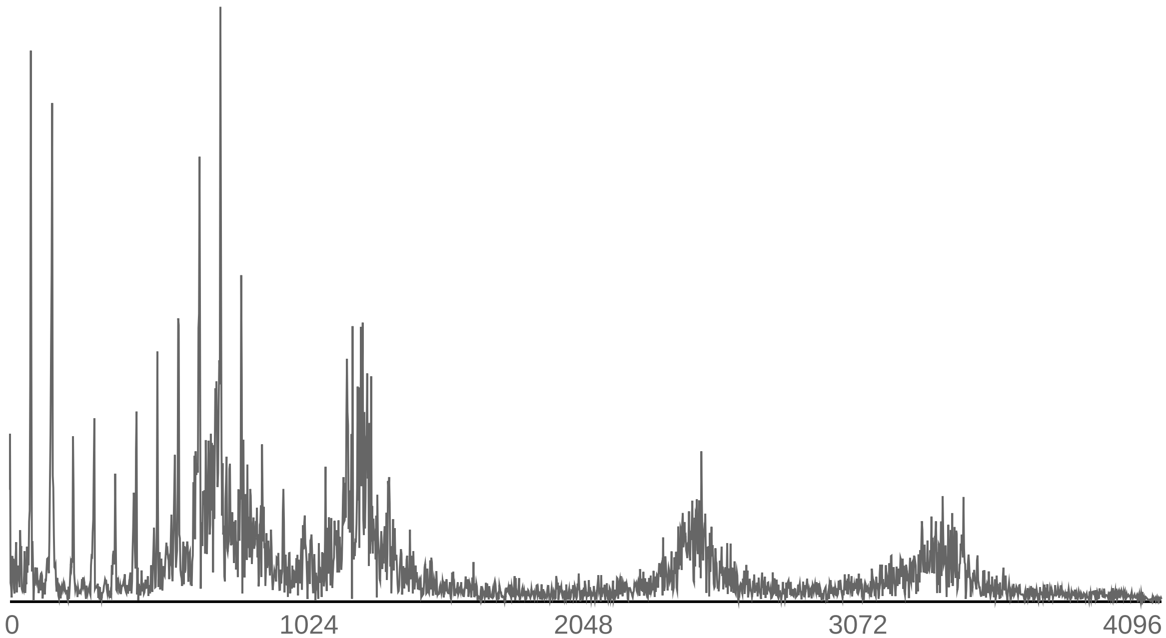
Chapter 16. Introduction to Fourier series: the frequency analysis of sound

The figure below is the graph of a recording of one-half second of my voice saying "Aaaaah."



This sound is not quite regular—it does not really repeat itself in any given period. Nonetheless, it is approximately periodic. We can make it into a periodic function simply by repeating this sound over and over again every half-second, and I think it is reasonable to guess that the result will not differ much in sound quality from the original, but just lasting longer. If we do make a periodic extension, we can then think about decomposing the sound into its simply periodic components. As we shall see, this can in fact be done in a manner conceptually easy but computationally hard. The approximate result is shown in this graph, which plots the intensity of sound in my voice recording versus frequency:



This graph shouldn't be taken too seriously, since I made the recording and the analysis in an afternoon. There are many interesting problems involved in doing this sort of thing better—the basic idea is that we should somehow process or **filter** the original data in order to get rid of artifacts and perform a more meaningful analysis. For example, I don't think the spiky lines at the bottom of the spectral graph are significant. Still, the general shape agrees with what *The fundamentals of musical acoustics* (by Arthur Benade), for example, says about the generic male human voice. There are some interesting small points—for example, when the the recording is played you can hear a definite wobble in the low range, which matches with the fuzziness of the spectral analysis at the low end. Another curious point to be made is that it is exactly the periodicity of sound which distinguishes vowels from consonants!

The fact that we can make a spectral decomposition of sounds is of enormous practical importance. The technique involved, called **Fourier analysis**, is part of the core in all fields of engineering and physics. It is fundamental not only in the analysis of sound, but in all parts of technology dealing with the transmission of information—including, for example, the technology of electronic visual display. The basic mathematics is relatively straightforward, as we shall see. The practical implementation is not quite so straightforward, but this I will only deal with briefly.

Fourier analysis can be applied to virtually any function, but in this part of the course we shall look only at periodic functions. Thus, for example, what we shall say here applies to vowel sounds but not to consonants. Fourier analysis can also be used with functions of several variables. This is what one requires to apply it to image processing or X-ray analysis.

1. A preview

The main mathematical result is that any reasonable periodic function with period T can be expressed as a sum of simply period functions:

- *If $f(t)$ is any reasonable function with $f(t+T) = f(t)$ for all t, then f can be expressed as an infinite linear combination*

$$f(t) = C_0 + \sum_{n>0} C_n \cos\left((2\pi nt/T) - \theta_n\right) .$$

It is remarkable that there is a simple way to calculate the coefficients C_n. For example, the coefficient C_0 is, as it has to be, the average of f over one period:

$$C_0 = \frac{1}{T}\int_0^T f(s)\,ds\ .$$

The formula for the other coefficients will be found by considering $f(t)e^{-2\pi imt/T}$ instead of $f(t)$ itself. The n-th term the series for $f(t)$ is often called the n-th mode or n-th **Fourier component** of $f(t)$. It is remarkable that in situations where $f(t)$ represents physical vibration of some kind, the energy of the motion being described by f is the sum of the energy of each of its components.