Problems in the Theory of Automorprhic Forms†

To Salomon Bochner
In Gratitude

1. There has recently been much interest, if not a tremendous amount of progress, in the arithmetic theory of automorphic forms. In this lecture I would like to present the views not of a number theorist but of a student of group representations on those of its problems that he finds most fascinating. To be more precise, I want to formulate a series of questions which the reader may, if he likes, take as conjectures. I prefer to regard them as working hypotheses. They have already led to some interesting facts. Although they have stood up for a fair length of time to the most careful scrutiny I could give, I am still not entirely easy about them. Indeed even at the beginning in the course of the definitions, which I want to make in complete generality, I am forced, for lack of time and technical competence, to make various assumptions.

I should perhaps apologize for such a speculative lecture. However, there are some interesting facts scattered amongst the questions. Moreover, the unsolved problems in group representations arising from the theory of automorphic forms are much less technical than the solved ones, and their significance can perhaps be more easily appreciated by the outsider.

Suppose G is a connected reductive algebraic group defined over a global field F. F is then an algebraic number field or a function field in one variable over a finite field. Let $\mathbb{A}(F)$ be the adèle ring of F. $G_{\mathbb{A}(F)}$ is a locally compact topological group with G_F as a discrete subgroup. The group $G_{\mathbb{A}(F)}$ acts on the functions on $G_F \setminus G_{\mathbb{A}(F)}$. In particular, it acts on $L^2(G_F \setminus G_{\mathbb{A}(F)})$. It should be possible, although I have not done so and it is not important at this stage, to attach a precise meaning ot the assertion that a given irreducible representation π of $G_{\mathbb{A}(F)}$ occurs in $L^2(G_F \setminus G_{\mathbb{A}(F)})$. If G is abelian it would mean that π is a character of $G_F \setminus G_{\mathbb{A}(F)}$. if G is not abelian it would be true for at least those representations which act on an irreducible invariant subspace of $L^2(G_F) \setminus G_{\mathbb{A}(F)}$.

If G is $GL(1)$ then to each such π one, following Hecke, associates an L-function. If G is $GL(2)$ then Hecke has also introduced, without explicitly mentioning group representations, some L-functions. The problems I want to discuss center about the possibility of defining L-functions for all such π and proving that they have the analytic properties we have grown used to expecting of such functions. I

† Appears in *Lectures in Modern Analysis and Applications* III, C. T. Taam, ed., Springer Lecture Notes in Mathematics 170, 1970, pp. 18–61

shall also comment on the possible relations of these new functions to the Artin L-functions and the L-functions attached to algebraic varieties.

Given G, I am going to introduce a complex analytic group $\hat{G}_F$. To each complex analytic representation σ of $\hat{G}_F$ and each π I want to attach an L-function $L(s,\sigma,\pi)$. Let me say a few words about the general way in which I want to form the function. $G_{\mathbb{A}(F)}$ is a restricted direct product $\coprod_{\mathfrak{p}} G_{F_\mathfrak{p}}$. The product is taken over the primes, finite and infinite, of F. It is reasonable to expect, although to my knowledge it has not yet been proved in general, that π can be represented as $\prod_{\mathfrak{p}} \otimes\pi_\mathfrak{p}$ where $\pi_\mathfrak{p}$ is a unitary representation of $G_{F_\mathfrak{p}}$.

I would like to have first associated to any algebraic group G defined over $F_\mathfrak{p}$ a complex analytic group $\hat{G}_{F_\mathfrak{p}}$ and to any complex analytic representation $\sigma_\mathfrak{p}$ of $\hat{G}_{F_\mathfrak{p}}$ and any unitary representation $\pi_\mathfrak{p}$ of $G_{F_\mathfrak{p}}$ a local L-function $L(s,\sigma_\mathfrak{p},\pi_\mathfrak{p})$ which, when $\mathfrak{p}$ is non-archimedean, would be of the form

$$\prod_{i=1}^{n} \frac{1}{1-\alpha_i|\varpi_\mathfrak{p}|^s}$$

where n is the degree of $\sigma_\mathfrak{p}$. Some of the α_i may be zero. For $\mathfrak{p}$ infinite it would be, basically, a product of Γ-functions. $L(s,\sigma_\mathfrak{p},\pi_\mathfrak{p})$ would depend only on the equivalence classes of $\sigma_\mathfrak{p}$ and $\pi_\mathfrak{p}$. I would also like to have defined for every non-trivial additive character $\psi_{F_\mathfrak{p}}$ of $F_\mathfrak{p}$ a factor $\varepsilon(s,\sigma_\mathfrak{p},\pi_\mathfrak{p},\psi_{F_\mathfrak{p}})$ which, as a function of s, has the form ae^{bs}.

There would be a complex analytic homomorphism of $\hat{G}_{F_\mathfrak{p}}$ into $\hat{G}_F$ determined up to an inner automorphism of $\hat{G}_F$. Thus σ determines for each $\mathfrak{p}$ a representation of $\sigma_\mathfrak{p}$ of $\hat{G}_{F_\mathfrak{p}}$. I want to define

$$L(s,\sigma,\pi) = \prod_{\mathfrak{p}} L(s,\sigma_\mathfrak{p},\pi_\mathfrak{p}) \ . \tag{A}$$

Of course it has to be shown that the product converges in a half-plane. We shall see how to do this. Then we will want to prove that the function can be analytically continued to a function meromorphic in the whole complex plane. Let ψ_F be a non-trivial character of $F\setminus\mathbb{A}(F)$ and let $\psi_{F_\mathfrak{p}}$ be the restriction of ψ_F to $F_\mathfrak{p}$. We will want $\varepsilon(s,\sigma_\mathfrak{p},\pi_\mathfrak{p},\psi_{F_\mathfrak{p}})$ to be 1 for all but finitely many $\mathfrak{p}$. We will also want

$$\varepsilon(s,\sigma,\pi) = \prod_{\mathfrak{p}} \varepsilon(s,\sigma_\mathfrak{p},\pi_\mathfrak{p},\psi_{F_\mathfrak{p}})$$

to be independent of ψ_F. The functional equation should be

$$L(s,\sigma,\pi) = \varepsilon(s,\sigma,\pi)L(1-s,\tilde{\sigma},\pi)$$

if $\tilde{\sigma}$ is the representation contragredient to σ.

We are asking for too much too soon. What we should try to do is to define the $L(s, \sigma_{\mathfrak{p}}, \pi_{\mathfrak{p}})$ and the $\varepsilon(s, \sigma_{\mathfrak{p}}, \pi_{\mathfrak{p}}, \psi_{F_{\mathfrak{p}}})$ when there is no ramification, verify that there is ramification at only a finite number of primes, and show that if the product in (A) is taken only over the unramified primes it converges for $Re\, s$ sufficiently large. As we learn how to prove the functional equation we shall be able to make the definitions at the unramified primes. By the way, we introduce the additive characters, whose appearance must appear rather mysterious, only because we can indeed prove some things and know better than to leave them out.

What does unramified mean in our context? First of all for $\mathfrak{p}$ to be unramified G will have to be quasi-split over $F_{\mathfrak{p}}$ and split over an unramified extension. In that case there is, as we shall see, a canonical conjugacy class of maximal compact subgroups of $G_{F_{\mathfrak{p}}}$. For $\mathfrak{p}$ to be unramified, the restriction of $\pi_{\mathfrak{p}}$ to any one of these groups will have to contain the identity representation. There is also a condition to be imposed on $\psi_{F_{\mathfrak{p}}}$. Although it is not very important I would like to mention it explicitly. If $\mathfrak{p}$ is non-archimedean the largest ideal of $F_{\mathfrak{p}}$ on which $\psi_{F_{\mathfrak{p}}}$ is trivial will have to be $O_{F_{\mathfrak{p}}}$, the ring of integers in $F_{\mathfrak{p}}$. If $F_{\mathfrak{p}}$ is $\mathbb{R}$ then $\psi_{F_{\mathfrak{p}}}(x)$ will have to be $e^{2\pi i x}$ and if $F_{\mathfrak{p}}$ is $\mathbb{C}$ then $\psi_{F_{\mathfrak{p}}}(z)$ will have to be $e^{4\pi i\, Re\, z}$. We want $\varepsilon(s, \sigma_{\mathfrak{p}}, \pi_{\mathfrak{p}}, \psi_{F_{\mathfrak{p}}})$ to be 1 if $\mathfrak{p}$ is unramified.

2. $\hat{G}_F$ can be identified for a connected reductive group over any field F. Take first a quasi-split group G over F which splits over the Galois extension K. Choose a Borel subgroup B of G which is defined over F and let T be a maximal torus of B which is also defined over F. Let L be the group of rational characters of T. Write G as G^0G^1 where G^0 is abelian and G^1 is semi-simple. Then $G^0 \cap G^1$ is finite. If $T^0 = G^0$ and $T^1 = T \cap G^1$ then $T = T^0T^1$. Let L^0_+ be the group of rational characters of T^0 and let L^0_- be the elements of L^0_+ which are 1 on $T^0 \cap T^1$. Let L^+_- be the group generated by the roots of T^1. If R is any field let $E^1_R = L^1_- \otimes_{\mathbb{Z}} R$. The Weyl group Ω acts on L^1_- and therefore on E^1_R. Let $(\,\cdot\,,\,\cdot\,)$ be a non-degenerate bilinear form on $E^1_{\mathbb{C}}$ which is invariant under Ω. Suppose also that its restriction to $E^1_{\mathbb{R}}$ is positive definite. Let

$$L^1_+ = \left\{ \lambda \in E^1_{\mathbb{C}} \;\middle|\; 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \text{ for all roots } \alpha \right\} .$$

Set $L_- = L^0_- \oplus L^1_-$ and $L_+ = L^0_+ \oplus L^1_+$. We may regard L as a sublattice of L_+. It will contain L_-.

Let $\alpha_1, \ldots, \alpha_\ell$ be the simple roots of T^1 with respect to B and let

$$(A_{ij}) = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$$

be the Cartan matrix. If σ belongs to $\mathfrak{G}(K/F)$ and λ belongs to L then $\sigma\lambda$, where $\sigma\lambda(t) = \sigma(\lambda(\sigma^{-1}t))$, also belongs to L. Thus $\mathfrak{G}(K/F)$ acts on L. It also acts on L_- and L_+ and the actions on these three lattices are consistent. Moreover the roots $\alpha_1, \dots, \alpha_\ell$ are permuted amongst themselves and the Cartan matrix is left invariant.

If R is any field containing $\mathbb{Q}$ let $E_R = L \otimes_{\mathbb{Z}} R$ and let $\hat{E}_R = \mathrm{Hom}_R(E_R, R)$. The lattices

$$\begin{aligned}\hat{L}_+ &= \mathrm{Hom}(L_-, \mathbb{Z}) = \mathrm{Hom}(L_-^0, \mathbb{Z}) \oplus \mathrm{Hom}(L_-^1, \mathbb{Z}) = L_+^0 \oplus \hat{L}_+^1 \\ \hat{L} &= \mathrm{Hom}(L, \mathbb{Z}) \\ \hat{L}_- &= \mathrm{Hom}(L_+, \mathbb{Z}) = \mathrm{Hom}(L_+^0, \mathbb{Z}) \oplus \mathrm{Hom}(L_+^1, \mathbb{Z}) = \hat{L}_-^0 \oplus \hat{L}_-^1\end{aligned}$$

may be regarded as subgroups of $\hat{E}_{\mathbb{C}}$. IF $E_R^0 = L_-^0 \otimes_{\mathbb{Z}} R$ then $E_R = E_R^0 \otimes E_R^1$. With the obvious definitions of $\hat{E}_R^0$ and $\hat{E}_R^1$ we have $\hat{E}_R = \hat{E}_R^0 \oplus \hat{E}_R^1$. Let $(\ \cdot\ ,\ \cdot\ )$ also denote the form on $\hat{E}_{\mathbb{C}}^1$ adjoint to the given form on $E_{\mathbb{C}}^1$. To be precise if λ and μ belong to $E_{\mathbb{C}}^1$, if $\hat{\lambda}$ and $\hat{\mu}$ belong to $\hat{E}_{\mathbb{C}}^1$, and if $\left\langle \eta, \hat{\lambda} \right\rangle = (\eta, \lambda)$ and $\langle \eta, \hat{\mu} \rangle = (\eta, \mu)$ for all η in $E_{\mathbb{C}}^1$ then $(\lambda, \mu) = (\hat{\lambda}, \hat{\mu})$.

If α is a root define its coroot $\hat{\alpha}$ in $\hat{E}_{\mathbb{C}}^1$ by the condition:

$$\langle \lambda, \hat{\alpha} \rangle = 2\,\frac{(\lambda, \alpha)}{(\alpha, \alpha)}$$

for all λ in $E_{\mathbb{C}}^1$. The coroots generate $\hat{L}_-^1$. Moreover

$$(\hat{\alpha}, \hat{\beta}) = 4\,\frac{(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)}$$

and

$$2\,\frac{(\hat{\alpha}, \hat{\beta})}{(\hat{\alpha}, \hat{\alpha})} = 2\,\frac{(\alpha, \beta)}{(\beta, \beta)}\ .$$

Thus the matrix

$$(\hat{A}_{ij}) = \left(2\,\frac{(\hat{\alpha}_i, \hat{\alpha}_j)}{(\hat{\alpha}_i, \hat{\alpha}_i)} \right)$$

is the transpose of (A_{ij}). The linear transformation $\hat{S}_i$ of $\hat{E}_{\mathbb{C}}^1$ defined by

$$\hat{S}_i(\hat{\alpha}_j) = \hat{\alpha}_j - \hat{A}_{ij}\hat{\alpha}_i = \hat{\alpha}_j - A_{ji}\hat{\alpha}_i$$

is contragredient to the linear transformation S_i of $\hat{E}_{\mathbb{C}}^1$ defined by

$$S_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i\ .$$

Thus the group $\hat{\Omega}$ generated by $\{\hat{S}_i | 1 \leq i \leq \ell\}$ is canonically isomorphic to the finite group Ω and, by a well-known theorem (cf. Chapter VII of [7]) $(\hat{A}_{ij})$ is the Cartan matrix of a simply-connected complex group $\hat{G}_+^1$. Let $\hat{B}_+^1$ be a Borel subgroup of $\hat{G}_+^1$ and let $\hat{T}_+^1$ be a Cartan subgroup in $\hat{B}_+^1$. We identify the simple roots of $\hat{T}_+^1$ with respect to $\hat{B}_+^1$ with $\hat{\alpha}_1, \dots, \hat{\alpha}_\ell$ and the free vector space over $\mathbb{C}$ with basis

$\{\hat{\alpha}_1, \ldots, \hat{\alpha}_\ell\}$ with $\hat{E}^1_{\mathbb{C}}$. We may also identify Ω and $\hat{\Omega}$. The roots of $\hat{T}^1_+$ are the vectors $\omega\hat{\alpha}_i$, $\omega \in \Omega$, $1 \le i \le \ell$. If $\omega\alpha_i = \alpha$ then $\omega\hat{\alpha}_i = \hat{\alpha}$ because

$$\langle \lambda, \omega\hat{\alpha}_i \rangle = \langle \omega^{-1}\lambda, \hat{\alpha}_i \rangle = 2\,\frac{(\omega^{-1}\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = 2\,\frac{(\lambda, \omega\alpha_i)}{(\omega\alpha_i, \omega\alpha_i)} = 2\,\frac{(\lambda, \alpha)}{(\alpha, \alpha)}$$

Thus the roots of $\hat{T}^1_+$ are just the coroots. If λ belongs to $\hat{E}^1_{\mathbb{C}}$ then

$$2\,\frac{(\lambda, \hat{\alpha})}{(\hat{\alpha}, \hat{\alpha})} = \langle \alpha, \lambda \rangle$$

so that

$$\hat{L}^1_+ = \left\{ \lambda \in \hat{E}^1_{\mathbb{C}} \;\middle|\; 2\,\frac{(\lambda, \hat{\alpha})}{(\hat{\alpha}, \hat{\alpha})} \in \mathbb{Z} \text{ for all coroots } \hat{\alpha} \right\}$$

and is therefore just the set of weights of $\hat{T}^1_+$.

Let

$$\hat{G}^0_+ = \mathrm{Hom}_{\mathbb{Z}}(\hat{L}^0_+, \mathbb{C}^*)\ .$$

$\hat{G}^0_+$ is a reductive complex Lie group. Set $\hat{G}_+ = \hat{G}^0_+ \times \hat{G}^1_+$. If $\hat{T}^0_+ = \hat{G}^0_+$ and $\hat{T}_+ = \hat{T}^0_+ \times \hat{T}^1_+$ then $\hat{L}_+$ is the set of complex analytic characters of $\hat{T}_+$. If

$$\hat{Z} = \left\{ t \in \hat{T}_+ \;\middle|\; \lambda(t) = 1 \text{ for all } \lambda \text{ in } \hat{L} \right\}$$

then $\hat{Z}$ is a normal subgroup of $\hat{G}_+$ and $\hat{G} = \hat{G}_+/\hat{Z}$ is also a complex Lie group. $\mathfrak{G}(K/F)$ acts in a natural fashion on $\hat{L}_-$, $\hat{L}$, and $\hat{L}_+$. The action leaves the set $\{\hat{\alpha}_1, \ldots \hat{\alpha}_\ell\}$ invariant. $\mathfrak{G}(K/F)$ acts naturally on $\hat{G}^0_+$. I want to define an action on $\hat{G}^1_+$ and therefore an action on $\hat{G}_+$. Choose $H_1, \ldots, H_\ell$ in the Lie algebra of $\hat{T}^1_+$ so that

$$\lambda(H_i) = \langle \alpha_i, \lambda \rangle$$

for all λ in $\hat{L}^1_+$. Choose root vectors $X_1, \ldots, X_\ell$ belonging to the coroots $\hat{\alpha}_1, \ldots, \hat{\alpha}_\ell$ and root vectors $Y_1, \ldots, Y_\ell$ belonging to their negatives. Suppose $[X_i, Y_i] = H_i$. If σ belongs to $\mathfrak{G}(K/F)$ let $\sigma(\hat{\alpha}_i) = \hat{\alpha}_{\sigma(i)}$. There is (cf. Chapter VII of [7]) a unique isomorphism σ of the Lie algebra of $\hat{G}^1_+$ such that

$$\sigma(H_i) = H_{\sigma(i)},\ \sigma(X_i) = X_{\sigma(i)},\ \sigma(Y_i) = Y_{\sigma(i)}\ .$$

These isomorphisms clearly determine an action of $\mathfrak{G}(K/F)$ on the Lie algebra and therefore one on $\hat{G}^1_+$ itself. Since $\mathfrak{G}(K/F)$ leaves L invariant its action on $\hat{G}_+$ can be transferred to $\hat{G}$. If $\hat{B}$ is the image of $\hat{B}_+ = \hat{T}^0_+ \times \hat{B}^1_+$ and $\hat{T}$ the image of $\hat{T}_+$ in $\hat{G}$ the action leaves $\hat{B}$ and $\hat{T}$ invariant. I want to define $\hat{G}_F$ to be the semi-direct product $\hat{G} \times \mathfrak{G}(K/F)$.

However $\hat{G}_F$ as defined depends upon the choice of B, T and $X_1 \dots, X_\ell$ and $\hat{G}_F$ comes provided with a Borel subgroup $\hat{B}$ of its connected component, a Cartan subgroup $\hat{T}$ of $\hat{B}$, and a one-to-one correspondence between the simple roots of T with respect to B and those of $\hat{T}$ with respect to $\hat{B}$. Suppose G' is another quasi-split group over F which is isomorphic to G over K by means of an isomorphism φ such that $\varphi^{-1}\sigma(\varphi)$ is inner for all σ in $\mathfrak{G}(K/F)$, B' is a Borel subgroup of G' defined over F, and T' is a Cartan subgroup of B' also defined over F. There is an inner automorphism ψ of G which is defined over K so that $\varphi\psi$ takes B to B' and T to T'. Then $\varphi\psi$ determines an isomorphism of $\hat{L}$ and $\hat{L}'$ and a one-to-one correspondence between $\{\alpha_1, \dots, \alpha_\ell\}$ and $\{\alpha'_1, \dots, \alpha'_\ell\}$ both of which depend only on φ and, as is easily verified, commute with the action of $\mathfrak{G}(K/F)$. There is then a natural isomorphism of $\hat{G}^0_+$ with $(\hat{G}^0_+)'$ associated to φ. Moreover there is a unique isomorphism of $\hat{G}^1_+$ with $(\hat{G}^1_+)'$ whose action on the Lie algebra takes H_i to H'_i, X_i to X'_i, and Y_i to Y'_i. The two together define an isomorphism of $\hat{G}_+$ with $\hat{G}'_+$. If we assume that α_i corresponds to α'_i, $1 \leq i \leq \ell$ this isomorphism takes $\hat{Z}$ to $\hat{Z}'$ and determines an isomorphism of $\hat{G}$ with $\hat{G}'$ which commutes with $\mathfrak{G}(K/F)$. This in turn determines an isomorphism $\hat{\varphi}$ of $\hat{G}'_F$ with $\hat{G}_F$. In particular taking $G' = G$ and φ to be the identity we see that $\hat{G}_F$ is determined up to a canonical isomorphism.

Suppose G is any reductive group over F, K is a Galois extension of F, G' and G'' are quasi-split groups over F which split over K, and φ: $G' \to G$, ψ: $G'' \to G$ are isomorphisms defined over K such that $\varphi^{-1}\sigma(\varphi)$ and $\psi^{-1}\sigma(\psi)$ are inner for all σ in $\mathfrak{G}(K/F)$. Then $(\psi^{-1}\varphi)^{-1}\sigma(\psi^{-1}\varphi)$ is also inner so that there is a canonical isomorphism of $\hat{G}'_F$ and $\hat{G}''_F$. We are thus free to set $\hat{G}_F = \hat{G}'_F$. $\hat{G}_F$ depends on K but there is no need to stress this. However we shall sometimes write $\hat{G}_{K/F}$ instead of $\hat{G}_F$.

3. Although it is a rather simple case, it may be worthwhile to carry out the previous construction when G is $GL(n)$ and $K = F$. We take T to be the diagonal and B to be the upper triangular matrices. G^0 is the group of non-zero scalar matrices and G^1 is $SL(n)$. If λ belongs to L and

$$\lambda\colon \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \longrightarrow t_1^{m_1} \dots t_n^{m_n}$$

with $m_1, \dots, m_n$ in $\mathbb{Z}$ we write $\lambda = (m_1, \dots, m_n)$. Thus L is identified with $\mathbb{Z}^n$. We may identify $E_\mathbb{R}$ with $\mathbb{R}^n$ and $E_\mathbb{C}$ with $\mathbb{C}^n$. If λ belongs to L^0_+ and

$$\lambda\colon tI \to t^m$$

with m in $\mathbb{Z}$ we write $\lambda = \left(\frac{m}{n}, \ldots, \frac{m}{n}\right)$. Then L_-^0 which is a subgroup of both L and L_+^0 consists of the elements $(m, \ldots, m)$ with m in $\mathbb{Z}$. The rank ℓ is $n-1$ and

$$\begin{aligned}
\alpha_1 &= (1, -1, 0, \ldots, 0) \\
\alpha_2 &= (0, 1, -1, 0, \ldots, 0) \\
&\vdots \\
\alpha_\ell &= (0, \ldots, 0, 1, -1)
\end{aligned}$$

Thus

$$L_-^1 = \left\{ (m_1, \ldots, m_n) \in L \;\Big|\; \sum_{i=1}^n m_i = 0 \right\} .$$

$E_{\mathbb{C}}^1$ is the set of all $(z_1, \ldots, z_n)$ in $E_{\mathbb{C}}$ for which

$$\sum_{i=1}^n z_i = 0 \ .$$

The bilinear form on $E_{\mathbb{C}}^1$ may be taken as the restriction of the form

$$(z, w) = \sum_{i=1}^n z_i w_i$$

on $E_{\mathbb{C}}$. Thus

$$L_+^1 = \left\{ (m_1, \ldots, m_n) \;\Big|\; \sum_{i=1}^n m_i = 0 \text{ and } m_i - m_j \in \mathbb{Z} \right\} .$$

We may use the given bilinear form to identify $\hat{E}_{\mathbb{C}}$ with $E_{\mathbb{C}}$. Then the operation " $\widehat{\ }$ " leaves all lattices and all roots fixed. Thus $\hat{G}_+^0 = \operatorname{Hom}(L_+^0, \mathbb{C})$. Any non-singular complex scalar matrix tI defines an element of $\hat{G}_+^0$, namely, the homomorphism

$$\left(\frac{m}{n}, \ldots, \frac{m}{n}\right) \longrightarrow t^m \ .$$

We identify $\hat{G}_+^0$ with the group of scalar matrices. $\hat{G}_+^1$ is $SL(n, \mathbb{C})$. There is a natural map of $\hat{G}_+^0 \times \hat{G}_+^1$ onto $GL(n, \mathbb{C})$. It sends $tI \times A$ to tA. The kernel is easily seen to be $\hat{Z}$ so that $\hat{G}_F$ is $GL(n, \mathbb{C})$.

4. To define the local L-functions, to prove that almost all primes are unramified, and to prove that the product of the local L-functions over the unramified primes converges for $Re\, s$ sufficiently large we need some facts from the reduction theory for groups over local fields (cf. [1]). Much progress has been made in that theory, but it is still incomplete. Unfortunately, the particular facts we need do not

seem to be in the literature. Very little is lost at this stage if we just assume them. For the groups about which something definite can be said, they are easily verified.

Suppose K is an unramified extension of the non-archimedean local field F and G is a quasi-split group over F which splits over K. Let B be a Borel subgroup of G and T a Cartan subgroup of B, both of which are defined over F. Let v be the valuation on K. It is a homomorphism from K^*, the multiplicative group of K, onto $\mathbb{Z}$ whose kernel is the group of units. If t belongs to T_F, let $v(t)$ in $\hat{L}$ be defined by $\langle \lambda, v(t) \rangle = v(\lambda(t))$ for all λ in L. If σ belongs to $\mathfrak{G}(K/F)$, then

$$\langle \lambda, \sigma v(t) \rangle = \left\langle \sigma^{-1}\lambda, v(t) \right\rangle = v(\sigma^{-1}(\lambda(\sigma t))) = v(\lambda(t))$$

because $\sigma t = t$ and $v(\sigma^{-1} a) = v(a)$ for all a in K^*. Thus v is a homomorphism of T_F into $\hat{M}$, the groups of invariants of $\mathfrak{G}(K/F)$ in $\hat{L}$. It is in fact easily seen that it takes T_F onto $\hat{M}$.

We assume the following lemma.

Lemma 1. *There is a Chevalley lattice in the Lie algebra of G whose stabilizer U_K is invariant under $\mathfrak{G}(K/F)$. U_K is its own normalizer. Moreover, $G_K = B_K U_K$, $H^1(\mathfrak{G}(K/F), U_K) = 1$, and $H^1(\mathfrak{G}(K/F), B_K \cap U_K) = 1$. If we choose two such Chevalley lattices with stabilizers U_K and U'_K, respectively, then U'_K is conjugate to U_K in G_K.*

If g belongs to G_K and σ belongs to $\mathfrak{G}(K/F)$, let $g^\sigma = \sigma^{-1}(g)$. If g belongs to G_F, we may write it as $= bu$ with b in B_K and u in U_K. Then $g^\sigma = b^\sigma u^\sigma$ and $u^\sigma u^{-1} = b^{-\sigma} b$. By the lemma, there is a v in $B_K \cap U_K$ such that $u^\sigma u^{-1} = b^{-\sigma} b = v^\sigma v^{-1}$. Then $b' = bv$ belongs to B_F, $u' = v^{-1} u$ belongs to $U_F = G_F \cap U_K$, and $g = b'u'$. Thus, $G_F = B_F U_F$.

If $g U_K g^{-1} = U'_K$ for some g in G_K, then $g^\sigma U_K g^{-\sigma} = U'_K$ so that $g^{-\sigma} g$ belongs to U_K, which is its own normalizer. By the lemma, there is u in U_K such that $g^{-\sigma} g = u^\sigma u^{-1}$. Then $g_1 = gu$ lies in G_F and $g_1 U_K g_1^{-1} = U'_K$. Thus, U_F and U'_F are conjugate in G_F.

Let $C_c(G_F, U_F)$ be the set of all compactly supported functions for G_F such that $f(gu) = f(ug) = f(g)$ for all u in U_F and all g in G_F. $C_c(G_F, U_F)$ is an algebra under convolution. It is called the Hecke algebra. If N is the unipotent radical of B let dn be a Haar measure on N_F and let $\frac{d(bnb^{-1})}{dn} = \delta(b)$ if b belongs to B_F. If λ belongs to $\hat{M}$, choose t in T_F such that $v(t) = \lambda$. If f belongs to $C_c(G_F, U_F)$, set

$$\hat{f}(\lambda) = \delta^{1/2}(t) \left\{ \int_{N_F \cap U_F} dn \right\}^{-1} \int_{N_F} f(tn)dn \ .$$

The group $\mathfrak{G}(K/F)$ acts on Ω. Let Ω^0 be the group of invariant elements. Ω^0 acts on $\hat{M}$. Let $A(\hat{M})$ be the group algebra of $\hat{M}$ over $\mathbb{C}$, and let $\Lambda^0(\hat{M})$ be the invariants of Ω^0 in $\Lambda(\hat{M})$. We also assume the following lemma (cf. [12]).

Lemma 2. *The map $f \to \hat{f}$ is an isomorphism of $C_c(G_F, U_F)$ and $\Lambda^0(M)$.*

Suppose B is replaced by B_1 and T by T_1. Observe that $T \simeq B/N$ and $T_1 \simeq B_1/N_1$. If u in G_F takes B to B_1, it takes N to N_1 and defines a map from T to T_1. This map does not depend on u. It determines $\mathfrak{G}(K/F)$ invariant maps from L_1 to L and from $\hat{L}$ to $\hat{L}_1$ and thus maps from $\hat{M}$ to $\hat{M}_1$ and from $\Lambda^0(\hat{M})$ to $\Lambda^0(\hat{M}_1)$. Suppose $\hat{f}$ goes to $\hat{f}_1$ and $\hat{\lambda}$ goes to $\hat{\lambda}_1$. If we choose, as we may, u in U_F, then

$$\hat{f}_1(\hat{\lambda}_1) = \hat{f}(\hat{\lambda}) = \delta^{1/2}(t)\left\{\int_{N_F \cap U_F} dn\right\}^{-1} \int_{N_F} f(tn)dn \ .$$

Let $N_F \cap U_F = V$. Denote the corresponding group associated to N_1 by V_1. Then $uVu^{-1} = V_1$. Choose $d(unu^{-1}) = dn_1$. Since $f(ugu^{-1}) = f(g)$, the expression on the right equals

$$\delta^{1/2}(utu^{-1})\left\{\int_{V_1} dn_1\right\}^{-1} \int_{N_F} f(utu^{-1}unu^{-1})dn \ .$$

If utu^{-1} projects on t_1 in T_1, then $\delta(utu^{-1}) = \delta(t_1)$ and $v(t_1) = \hat{\lambda}_1$. Moreover,

$$\int f(utu^{-1}unu^{-1})dn = \int f(t_1n_1)dn_1$$

and the diagram

$$\begin{array}{ccc} & C_c(G_F, U_F) & \\ \swarrow & & \searrow \\ \Lambda^0(M) & \longrightarrow & \Lambda^0(M_1) \end{array}$$

is commutative.

If $gU_Fg^{-1} = U'_F$, the map $f \to f'$ with $f'(h) = f(g^{-1}hg)$ is an isomorphism of $C_c(G_F, U_F)$ with $C_c(G_F, U'_F)$. It does not depend on g. We can take g in B_F. Then

$$\hat{f}'(\lambda) = \delta^{1/2}(t)\left\{\int_{N_F \cap U'_F} dn\right\}^{-1} \int_{N_F} f(g^{-1}tng)dn \ .$$

Since $g^{-1}tng = t(t^{-1}g^{-1}tg)g^{-1}ng$, the second integral is equal to

$$\int_{N_F} f(tg^{-1}ng)dn \ .$$

Since

$$\frac{d(g^{-1}ng)}{dn} = \left\{\int_{N_F \cap U'_F} dn\right\}^{-1} \int_{N_F \cap U_F} dn$$

we conclude that $\hat{f}'(\hat{\lambda}) = \hat{f}(\hat{\lambda})$ and that the diagram

$$\begin{array}{ccc} C_c(G_F, U_F) & \longrightarrow & C_c(G_F, U'_F) \\ & \searrow \quad \swarrow & \\ & \Lambda^0(\hat{M}) & \end{array}$$

is commutative.

I shall not explicitly mention the commutativity of these diagrams again. However, they are important because they imply that the definitions to follow have the invariance properties which are required if they are to have any sense.

If π is an irreducible unitary representation of G_F on H whose restriction to U_F contains the identity representation, then

$$H_0 = \{x \in H \bigm| \pi(u)x = x \text{ for all } u \text{ in } U_F\}$$

is a one-dimensional subspace. If f belongs to $C_c(G_F, U_F)$, then

$$\pi(f) = \int_G f(g)\pi(g)dg$$

maps H_0 into itself. The representation of $C_c(G_F, U_F)$ on H_0 determines a homomorphism χ of $C_c(G_F, U_F)$ or of $\Lambda^0(\hat{M})$ into the ring of complex numbers. π is determined by χ. To define the local L-functions, we study such homomorphisms. First of all, observe that, if χ is associated to a unitary representation, then

$$|\chi(f)| \le \int_{G_F} |f(g)|dg\ .$$

Since $\Lambda(\hat{M})$ is a finitely generated module over $\Lambda^0(\hat{M})$, any homomorphism of $\Lambda^0(\hat{M})$ into $\mathbb{C}$ may be extended to a homomorphism of $\Lambda(\hat{M})$ into $\mathbb{C}$ which will necessarily be of the form

$$\Sigma\hat{f}(\lambda)\lambda \longrightarrow \Sigma\hat{f}(\lambda)\lambda(t) \tag{B}$$

for some t in $\hat{T}$. Conversely, given t the formula (B) determines a homomorphism χ_t of $\Lambda^0(\hat{M})$ into $\mathbb{C}$. We shall show that $\chi_{t_1} = \chi_{t_2}$ if and only if $t_1 \times \sigma_F$ and $t_2 \times \sigma_F$, where σ_F is the Frobenius subsitution, are conjugate in $\hat{G}_F$. If t belongs to $\hat{G}$ and σ belongs to $\mathfrak{G}(K/F)$, we shall abbreviate $t \times \sigma$ to $t\sigma$. It is known [4] that every semi-simple element of $\hat{G}_F$ whose projection on $\mathfrak{G}(K/F)$ is σ_F is conjugate to some $t\sigma_F$ with t in $\hat{T}$. Thus, there is a one-to-one correspondence between homomorphisms of the Hecke algebra into $\mathbb{C}$ and semi-simple conjugacy classes in $\hat{G}_F$ whose projection on $\mathfrak{G}(K/F)$ is σ_F.

If ρ is a complex analytic representation of $\hat{G}_F$ and χ_t is the homomorphism of $\Lambda^0(\hat{M})$ into $\mathbb{C}$ associated to π, we define the local L-function to be

$$L(s, \rho, \pi) = \frac{1}{\det(I - \rho(t\sigma_F)|\pi_F|^2)}$$

if π_F generates the maximal ideal of O_F.

$\hat{T}$ may be identified with $\mathrm{Hom}_{\mathbb{Z}}(\hat{L}, \mathbb{C}^*)$. The exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\psi} \mathbb{C}^* \longrightarrow 0$$

with $\varphi(z) = \frac{2\pi i}{\log|\pi_F|} z$ and $\psi(z) = |\pi_F|^{-z}$ leads to the exact sequence

$$0 \longrightarrow L = \mathrm{Hom}_{\mathbb{Z}}(\hat{L}, \mathbb{Z}) \xrightarrow{\varphi} E_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(\hat{L}, \mathbb{C}) \xrightarrow{\psi} \hat{T} \longrightarrow 0 \ .$$

Let $V_{\mathbb{C}}$ be the invariants of $\mathfrak{G}(K/F)$ in $E_{\mathbb{C}}$ and let $W_{\mathbb{C}}$ be the range of $\sigma_F - 1$. Then $E_{\mathbb{C}} = V_{\mathbb{C}} \oplus W_{\mathbb{C}}$. If w belongs to $W_{\mathbb{C}}$ and λ belongs to $\hat{M}$, then $\langle w, \lambda \rangle = 0$ and, replacing t by $t\psi(w)$ does not change χ_t. If $w = \sigma_F v - v$ and $\psi(v) = s$, then

$$t\psi(w)\sigma_F = ts^{-1}\sigma_F(s)\sigma_F = s^{-1}(t\sigma_F)s$$

is conjugate to $t\sigma_F$. Thus, we have to show that if $t_1 = \psi(v_1)$ and $t_2 = \psi(v_2)$ with v_1 and v_2 in $\mathbb{C}$, then $t_1\sigma_F$ and $t_2\sigma_F$ are conjugate if and only if $\chi_{t_1} = \chi_{t_2}$.

Some preliminary remarks are necessary. We also have a decomposition $\hat{E}_{\mathbb{C}} = \hat{V}_{\mathbb{C}} \oplus \hat{W}_{\mathbb{C}}$ and $\hat{M} = \hat{L} \cap \hat{V}_{\mathbb{C}}$. Let $\hat{Q}$ be the elements of $\hat{V}_{\mathbb{C}}$ obtained by projecting the positive coroots on $\hat{V}_{\mathbb{C}}$. If S is an orbit of $\mathfrak{G}(K/F)$ in the set of positive coroots, every element in S has the same projection on $\hat{V}_{\mathbb{C}}$. Since $\Sigma_{\hat{\alpha}\in S}\hat{\alpha}$ belongs to $\hat{V}_{\mathbb{C}}$, the projection must be

$$\frac{1}{n(S)}\Sigma_{\hat{\alpha}\in S}\hat{\alpha}$$

if $n(S)$ is the number of elements in S. Let $S_1, \ldots, S_m$ be the orbits of $\mathfrak{G}(K/F)$ in $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_\ell\}$ and set

$$\hat{\beta}_i = \frac{1}{n(S_i)}\Sigma_{\hat{\alpha}\in S_i}\hat{\alpha} \ .$$

Every element of $\hat{Q}$ is a linear combination of $\hat{\beta}_1, \ldots, \hat{\beta}_m$ with non-negative coefficients. Notice that if ω belongs to Ω^0 and ω acts trivially on $\hat{M}$, then ω leaves each β_i fixed and therefore takes positive roots to positive roots. Thus, it is 1. If we extend the inner product in any way from $\hat{E}^1_{\mathbb{R}}$ to $\hat{E}_{\mathbb{R}}$ and set

$$\hat{C} = \left\{x \in \hat{V}_{\mathbb{R}} \mid (\hat{\beta}_i, x) \geq 0,\ 1 \leq i \leq m\right\}$$

and

$$\hat{D} = \left\{x \in \hat{E}_{\mathbb{R}} \mid (\hat{\alpha}_i, x) \geq 0,\ 1 \leq i \leq \ell\right\} \ ,$$

then $\hat{C} = \hat{D} \cap \hat{V}_{\mathbb{R}}$. Consequently, no two elements of $\hat{C}$ belong to the same orbit of Ω^0.

Let $\widehat{\mathfrak{g}}_i$ be the subalgebra of the Lie algebra of $\hat{G}$ generated by the root vectors belonging to the coroots in S_i and their negatives. $\widehat{\mathfrak{g}}_i$ is fixed by $\mathfrak{G}(K/F)$. Let $\hat{G}_i$ be the corresponding analytic group and let $\hat{T}_i = \hat{T} \cap \hat{G}_i$. Let μ_i be the unique element of the Weyl group of $\hat{T}_i$ which takes every positive root to a negative root. If σ belongs to $\mathfrak{G}(K/F)$, then $\sigma(\mu_i)$ has the same property, so that $\sigma(\mu_i) = \mu_i$. Let w be any element in the normalizer of $\hat{T}$ whose image in $\hat{\Omega}$ is μ_i. Then $w\sigma_F(w^{-1})$ lies in $\hat{T}$. Its image in $\hat{T}/\psi(W_{\mathbb{C}})$ is independent of w. I claim that this image is 1. To see this write $\widehat{\mathfrak{g}}_i$ as a direct sum $\sum_{k=1}^{n} \widehat{\mathfrak{g}}_{ik}$ of simple algebras. If $[K\colon F] = n$ the stabilizer of $\widehat{\mathfrak{g}}_{i1}$ is $\left\{\sigma_F^{jn_i} \;\middle|\; 0 \le j \le \frac{n}{n_i}\right\}$. We may suppose that

$$\widehat{\mathfrak{g}}_{ik} = \sigma_F^{k-1}\left(\widehat{\mathfrak{g}}_{i1}\right) \ .$$

If $\hat{G}_{ik}$ is the analytic subgroup of $\hat{G}$ with Lie aglebra $\widehat{\mathfrak{g}}_{ik}$, choose w_1 in the normalizer of $\hat{T} \cap \hat{G}_{i1}$ so that w_1 takes the positive roots of $\widehat{\mathfrak{g}}_{i1}$ to the negative roots. We may choose w to be $\prod_{k=0}^{n_i-1} \sigma_F^k(w_1)$. Then

$$\begin{aligned} w\sigma_F(w^{-1}) &= (w_1\sigma_F(w_1^{-1}))(\sigma_F(w_1)\sigma_F^2(w_1^{-1}))\ldots(\sigma_F^{n_i-1}(w_1)\sigma_F^{n_i}(w_1^{-1})) \\ &= w_1\sigma_F^{n_i}(w_1^{-1}) \ . \end{aligned}$$

The Dynkin diagram of $\widehat{\mathfrak{g}}_{i1}$ is connected and the stabilizer of $\widehat{\mathfrak{g}}_{i1}$ in $\mathfrak{G}(K/F)$ acts transitively on it. This means that it is of type A_1 or A_2.

In the first case the diagram reduces to a point and the action of the stabilizer must be trivial, so that $w_1 = \sigma_F^{n_i}(w_1)$. In the second case $SL(3,\mathbb{C})$ is the simply-connected covering group of G_{i1}; we may choose the covering map to be such that $\hat{T} \cap \hat{G}_{i1}$ is the image of the diagonal matrices and $\sigma_F^{n_i}$ corresponds to the automorphism

$$A \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \, {}^tA^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

of $SL(3,\mathbb{C})$. We may take w_1 to be the image of

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ .$$

Then $\sigma_F^{n_i}(w_1) = w_1$.

μ_i acts on $\hat{V}$ as the reflection in the hyperlane perpendicular to β_i. Thus $\mu_1, \ldots, \mu_m$ generate Ω^0. If ω belongs to Ω^0, choose w in the normalizer of $\hat{T}$ whose image in Ω is ω. The image of $w\sigma_F(w^{-1})$ in $\hat{T}/\psi(W_{\mathbb{C}})$ depends only on ω. Call it δ_ω. Then

$$\delta_{\omega_1\omega_2} = w_1w_2\sigma_F(w_2^{-1}w_1^{-1}) = w_1(w_2\sigma_F(w_2^{-1}))w_1^{-1}(w_1\sigma_F(w_1^{-1})) = \omega_1(\delta_{\omega_1})\delta_{\omega_1} \ .$$

Since δ_ω is 1 on a set of generators, this relation shows that it is identically 1.

Returning to the original problem, we show first that if $\chi_{t_1} = \chi_{t_2}$ there is an ω in Ω^0 such that $\omega(w_1) = t_2$. Then, if w lies in the normalizer of $\hat{T}$ in $\hat{G}$ and its image in Ω is ω, we will have $w(t_1\sigma_F)w^{-1} = t_2 w\sigma_F(w^{-1})\sigma_F$. Since $w\sigma_F(w^{-1})$ lies in $\psi(W_{\mathbb{C}})$, the element on the right is conjugate to $t_2\sigma_F$.

If t belongs to $\hat{T}$, let χ_t also denote the homomorphism

$$\Sigma\hat{f}(\lambda)\lambda \to \Sigma\hat{f}(\lambda)\lambda(t)$$

of $\Lambda(\hat{M})$ into $\mathbb{C}$. If there were no ω such that $\omega(t_1) = t_2$, there would be an $\hat{f}$ in $\Lambda(\hat{M})$ such that

$$\chi_{t_2}(\hat{f}) \neq \chi_{\omega(t_1)}(\hat{f})$$

for all ω in Ω^0. Let

$$\prod(X - \omega(\hat{f})) = \sum_{k=0}^{n} \hat{f}_k X^k \ .$$

Each $\hat{f}_k$ belongs to $\Lambda^0(\hat{M})$. Applying χ_{t_1} and χ_{t_2}, we find that

$$\prod_{\omega}(X - \chi_{\omega(t_1)}(\hat{f})) = \sum_{k=0}^{n} \chi_{t_1}(\hat{f}_k)X^k = \sum_{k=0}^{n} \chi_{t_2}(\hat{f}_k)X^k = \prod_{\omega}(X - \chi_{\omega(t_2)}(\hat{f})) \ .$$

The polynomial on the right has $\chi_{t_2}(\hat{f})$ as a root, but that on the left does not. This is a contradiction.

If $t_1\sigma_F$ and $t_2\sigma_F$ are conjugate, then for every representation ρ of $\hat{G}_F$

$$\text{trace } \rho(t_1\sigma_F) = \text{trace } \rho(t_2\sigma_F) \ .$$

Let ρ act on X and if λ belongs to $\hat{M}$, let t_λ be the trace of $\rho(\sigma_F)$ on

$$X_\lambda = \left\{x \in X \ \middle|\ \rho(t)x = \lambda(t)x \text{ for all } t \text{ in } \hat{T}\right\} \ .$$

If t belongs to $\psi(W_{\mathbb{C}})$, then $\lambda(t) = 1$. If ω belongs to Ω^0 and w in the normalizer of T has image ω in Ω, the $X_{\omega\lambda} = \rho(w)X_\lambda$. Then $t_{\omega\lambda}$ is the trace of $w^{-1}\sigma_F w = w^{-1}\sigma_F(w)\sigma_F$ on X_λ. Since $\lambda(w^{-1}\sigma_F(w)) = 1$, we have $t_{\omega\lambda} = t_\lambda$ and

$$\text{trace } \rho(t\sigma_F) = \Sigma_{\lambda\in\hat{C}} t_\lambda \left(\Sigma_{\mu\in S(\lambda)}\mu(t)\right)$$

if $S(\lambda)$ is the orbit of λ. If

$$\hat{f}_\rho = \Sigma_{\lambda\in\hat{C}} t_\lambda \Sigma_{\mu\in S(\lambda)}\mu$$

then $\hat{f}_\rho$ belongs to $\Lambda^0(\hat{M})$ and

$$\text{trace } \rho(t\sigma_F) = \chi_t(\hat{f}_\rho) \ .$$

All we need to show is that the elements $\hat{f}_\rho$ generate $\Lambda^0(\hat{M})$ as a vector space. This is an easy induction argument because every λ in $\hat{C}$ is the highest weight of a representation of $\hat{G}_F$ whose restriction to $\hat{G}$ is irreducible.

5. If t belongs to $\hat{T}$, there is a unique function ϕ_t on G_F which satisfies $\phi_t(ug) = \phi_t(gu) = \phi_t(g)$ or all u in U_F and all g in $_F$, such that

$$\chi_t(f) = \int_{G_F} \phi_t(g) f(g) dg$$

for all f in $C_c(G_F, U_F)$. A formula for ϕ_t, valid under very general assumptions, has been found by I. G. MacDonald. However, because of the present state of reduction theory, his assumptions do not cover the cases in which we are interested. I am going to assume that the obvious generalization of his theorem is valid. In stating it we may as well suppose that t belongs ot $\psi(V_{\mathbb{C}})$.

Let $\hat{N}$ be the unipotent radical of $\hat{B}$, let $\hat{\mathfrak{n}}$ be its Lie algebra, and let τ be the representation of $\hat{T} \times \mathfrak{G}(K/F)$ on $\hat{\mathfrak{n}}$. If t belongs to $\psi(V_{\mathbb{C}})$, consider the function θ_t on $\hat{M}$ defined by

$$\theta_t(\lambda) = c|\pi_F|^{-\langle \rho, \lambda \rangle} \sigma_{\omega \in \Omega^0} \frac{\det(I - |\pi_F| \tau^{-1}(\omega(t)\sigma_F))}{\det(I - \tau^{-1}(\omega(t)\sigma_F))} \lambda^{-1}(\omega(t)) \ .$$

If $n(\hat{B})$ is the number of positive roots projection onto $\hat{\beta}$ in $\hat{Q}$,

$$c = \prod_{\beta \in Q} \left\{ \frac{1 - |\pi_F|^{n(\hat{\beta}) \langle \rho, \hat{\beta} \rangle}}{1 - |\pi_F|^{n(\beta)(\langle \rho, \beta \rangle + 1)}} \right\} \ .$$

As it stands, $\theta_t(\lambda)$ makes sense only when none of the eigenvalues of $\tau(\omega(t)\sigma)F$ are 1 for any ω in Ω^0. However, using the results of Kostant [8], we can write it in a form which makes sense for all t. Let $\hat{\rho}$ be one-half the sum of the positive coroots. $\hat{\rho}$ belongs to $\hat{V}$. If λ belongs to $\hat{M}$ and $\lambda + \hat{\rho}$ is non-singular, that is $(\lambda + \hat{\rho}, \hat{\beta}) \neq 0$ for all $\hat{\beta}$ in $\hat{Q}$, let ω in Ω^0 take $\lambda + \hat{\rho}$ to $\hat{C}$ and let χ_λ be $\operatorname{sgn} \omega$ times the character of the representation of $\hat{G}_F$ with highest weight $\omega(\lambda + \hat{\rho}) - \hat{\rho}$. If $\lambda + \hat{\rho}$ is singular, let $\chi_\lambda \equiv 0$. If

$$\det(I - |\pi_F| \tau^{-1}(t\sigma_F)) = \Sigma_{\mu \in \hat{M}} b_\mu \mu(t)$$

then

$$\theta_t(\lambda) = c|\pi_F|^{-\langle \rho, \lambda \rangle} \Sigma_{\mu \in \hat{M}} b_\mu \chi_{\mu - \lambda}((t\sigma_F)) \ .$$

Clearly $b_\mu$is 0 unless

$$\mu = -\Sigma_{\hat{\alpha} \in S} \hat{\alpha}$$

where S is a subset of the set of positive coroots invariant under $\mathfrak{G}(K/F)$. If U is the collection of such μ, then $\{\hat\rho+\mu|\mu\in M\}$ is invariant under Ω^0. Suppose $\hat\rho+\mu$ is non-singular and belongs to $\hat C$. Since $\langle\alpha_i,\hat\rho\rangle=1$ and $\langle\alpha_i,\mu\rangle$ is integral, for $1\le i\le\ell$, μ itself must belong to $\hat C$. This can only happen if μ is 0. Thus if $b_\mu\neq 0$ either $\hat\rho+\mu$ is singular or $\hat\rho+\mu$ belongs to the orbit of $\hat\rho$ and $\chi_\mu(g)\equiv\pm1$ on $\hat G_F$. As a consequence $\theta_t(0)$ is independent of t. Choose t_0 such that $\hat\beta_i(t_0)=|\pi|^{-\langle\rho,\hat\beta_i\rangle}$ for $1\le i\le m$. The eigenvalues of $\tau(\omega(t_0)\sigma_F)$ are the numbers $\zeta|\pi_F|^{-\langle\rho,\omega^{-1}\hat\beta\rangle}$ where $\hat\beta$ belongs to Q and ζ is an $n(\hat\beta)^{\text{th}}$ root of unity. If $\omega\neq1$ there is a $\hat\beta_i$ such that $\omega^{-1}\hat\beta=-\hat\beta_i$ for some β in Q. Then $\left\langle\rho,\omega^{-1}\hat\beta\right\rangle=-\left\langle\rho,\hat\beta_i\right\rangle=-1$ and $\tau(\omega(t_0)\sigma_F)$ has $|\pi_F|$ is an eigenvalue. Thus

$$\theta_{t_0}(0)=c\,\frac{\det(I-|\pi_f|\tau^{-1}(t_0\sigma_F))}{\det(I-\tau^{-1}(t_0\sigma_F))}=1\ .$$

We are going to assume that if t belongs to $\psi(V_{\mathbb C})$, a belongs to T_F, and $\lambda=v(a)$, then

$$\phi_t(a)=\theta_t(\lambda)\ .$$

If

$$|\chi_t(f)|\le\int_{G_F}|f(g)|dg$$

for all f in $C_c(G_F,U_F)$ then ϕ_t is bounded. I want to show that if ϕ_t is bounded, λ belongs to $\hat L$, $\bar\lambda$ in $\hat D$ belongs to the orbit of λ under Ω, and t lies in $\psi(V_{\mathbb C})$, then

$$|\lambda(t)|\le|\pi_F|^{-\langle\rho,\bar\lambda\rangle}\ .$$

Let $t=\psi(v)$. Then v is not determined by t but $Re\,v$ is and

$$|\lambda(t)|=|\pi_F|^{-Re\langle v,\lambda\rangle}\ .$$

We will show that if ϕ_t is bounded then $Re\,\langle v,\lambda\rangle\le\left\langle\rho,\bar\lambda\right\rangle$ for all λ in $\hat E_{\mathbb R}$. If ω belongs to Ω^0 and $Re\,\omega v$ lies in $\hat C$ then $Re\,\langle\omega v,\omega\lambda\rangle=Re\,\langle v,\lambda\rangle$. With no loss of generality, we may suppose that v lies in C, the analogue of $\hat C$. Then, as is well-known,

$$Re\,\langle v,\lambda\rangle\le Re\left\langle v,\bar\lambda\right\rangle$$

and we may as well assume that $\lambda=\bar\lambda$. We want to show that $Re\,\langle v,\lambda\rangle\le\langle\rho,\lambda\rangle$ for all λ in $\hat D$. Since ρ and v both belong to $V_{\mathbb C}$, it is sufficient to verify it for λ in $\hat C$. Let $\hat C^0$ be the interior of $\hat C$. The set of λ in $\hat C$ for which the assertion is true is closed, convex, and positively homogeneous. Therefore, if it contains $\hat M\cap\hat C^0$, it is $\hat C$.

Let S be the set of simple coroots $\hat{\alpha}$ for which $Re\,\langle v, \hat{\alpha}\rangle = 0$. Let Σ_0 be the positive coroots which are linear combinations of the elements of S and let Σ_+ be the other positive coroots. If $\hat{\mathfrak{n}}_0$ is the span of the root vectors associated to the coroots in Σ_0 and $\hat{\mathfrak{n}}_+$ is the span of the root vectors associated to the coroots in Σ_+, then τ breaks up into the direct sum of a representation τ_0 on $\hat{\mathfrak{n}}_0$ and a representation τ_+ on $\hat{\mathfrak{n}}_+$. Let $\hat{H}$ be the analytic subgroup of $\hat{G}_F$ whose Lie algebra is generated by the root vectors associated to the coroots of Σ_0 and their negatives and let Θ^0 be the subgroup of Ω^0 consisting of those elements with representatives in $\hat{H}$. If ω belongs to Ω^0 and $Re\,\omega v = Re\,v$, then ω belongs to Θ^0. If $Re\,\omega v \neq Re\,v$, then $Re\,\langle \omega v, \lambda\rangle < Re\,\langle v, \lambda\rangle$ for λ in $\hat{M} \cap \hat{C}^0$. Write $\lambda = \lambda_1 + \lambda_2$ where λ_1 is a linear combination of the coroots in S and λ_2 is orthogonal to these roots. If $s = \psi(u)$ with u in $V_{\mathbb{C}}$, consider

$$\theta_s'(\lambda) = c|\pi_F|^{\langle u-\rho,\lambda_2\rangle}\frac{\det(I - |\pi_F|\tau_+^{-1}(s\sigma_F))}{\det(I - \tau_+^{-1}(s\sigma_F))}\left\{\sum_{\Theta^0}\frac{\det(I - |\pi_F|\tau_0^{-1}(s\sigma_F))}{\det(I - \tau_0^{-1}(s\sigma_F))}|\pi_F|^{\langle \omega u-\rho,\lambda_1\rangle}\right\} .$$

The function θ_s' is not necessarily defined for all s. However, the preceding discussion, applied to $\hat{H}$ rather than $\hat{G}$, shows that it is defined at t and that $\theta_t'(0) \neq 0$. A simple application of l'Hospital's rule shows that, as a function of λ, θ_t' is the product of $|\pi_F|^{\langle v-\rho,\lambda\rangle}$ and a linear combination of products of polynomials and purely imaginary exponentials in λ_1. Thus, it does not vanish identically in any open cone.

Set $\theta_t'' = \theta_t - \theta_t'$. θ_t'' is a linear combination of products of polynomials in λ and an exponential $|\pi_F|^{\langle \omega v-\rho,\lambda\rangle}$ with $Re\,\omega v \neq Re\,v$. Thus, if λ belongs to the interior of $\hat{C}$,

$$\lim_{n\to-\infty} |\pi_F|^{\langle \rho-v,n\lambda\rangle}\theta_t''(n\lambda) = 0$$

and

$$\lim_{n\to-\infty} |\pi_F|^{\langle \rho-v,n\lambda\rangle}\theta_t(n\lambda) = \lim_{n\to-\infty} |\pi_F|^{\langle \rho-v,n\lambda\rangle}\theta_t'(n\lambda) .$$

If $\langle \rho, \lambda\rangle$ is less than $Re\,\langle v, \lambda\rangle$ for some λ in $\hat{C}$, then $\langle \rho, \lambda\rangle$ is less than $Re\,\langle v, \lambda\rangle$ for a λ in $\hat{C}$ for which $\theta_t'(n\lambda)$ does not vanish identically as a function of n. Since ϕ_t is bounded,

$$\lim_{n\to-\infty} |\pi_F|^{\langle \rho-v,n\lambda\rangle}\theta_t'(n\lambda) = 0 .$$

But $|\pi_F|^{\langle \rho-v,n\lambda\rangle}\theta_t'(n\lambda)$ is a function of the form

$$\sum_{k=0}^{q} \varphi_k(n)n^k$$

where $\varphi_k(n)$ is a linear combination of purely imaginary exponentials e^{ixn}. It is easy to see that it cannot approach 0 as n approaches $-\infty$.

6. Suppose G is a group defined over the global field F. There is a quasi-split group G' over F and an isomorphism φ: $G \to G'$ defined over a Galois extension K of F such that, for every σ in $\mathfrak{G}(K/F)$, $a_\sigma = \varphi^\sigma \varphi^{-1}$ is an inner automorphism of G'. We assume that there is a lattice $\mathfrak{g}_{O_F}$ over O_F in the Lie algebra of G' such that $O_K \mathfrak{g}_{0_F}$ is a Chevalley lattice.

If $\mathfrak{p}$ is a finite prime of K and $\mathfrak{P}$ is a prime of K dividing $\mathfrak{p}$, the group G over $F_\mathfrak{p}$ is obtained from G' by twisting by the restriction $\overline{a}$ of the cocycle $\{a_\sigma\}$ to $\mathfrak{G}(K_\mathfrak{P}/F_\mathfrak{p})$. Let $\overline{G}'$ be the adjoint group of G'. If $\bar{U}'_{K_\mathfrak{P}}$ is the stabilizer of the lattice $O_{K_\mathfrak{P}} \mathfrak{G}_{0_F}$ then, for almost all $\mathfrak{p}$, $\overline{a}$ takes values in $\bar{U}'_{K_\mathfrak{P}}$. If $K_\mathfrak{P}/F_\mathfrak{p}$ is also unramified, then G is quasi-split over $F_\mathfrak{p}$ because $H^1(\mathfrak{G}(K_\mathfrak{P}/F_\mathfrak{p}), \bar{U}'_{K_\mathfrak{p}}) = \{1\}$. Let S be the set of those $\mathfrak{p}$, unramified in K, for which $\overline{a}$ takes values in $\bar{U}'_{K_\mathfrak{P}}$. Let G act on a vector space X over F and let X_{O_F} be a lattice in X_F. Let $U_{F_\mathfrak{p}}$ be the stabilizer of $O_{F_\mathfrak{p}} X_{0_F}$ in $G_{F_\mathfrak{p}}$ and let $U'_{F_\mathfrak{p}}$ be the stabilizer of $O_{F_\mathfrak{p}} \mathfrak{g}_{O_F}$ in $G'_{F_\mathfrak{p}}$. Then $\varphi(U_{F_\mathfrak{p}}) = U'_{F_\mathfrak{p}}$ for almost all $\mathfrak{p}$. If $\mathfrak{p}$ is also in S, choose u in $\bar{U}'_{F_\mathfrak{p}}$ so that $\varphi^\sigma \varphi^{-1} = \mathrm{Ad}\, u^\sigma u^{-1}$ for all σ in $\mathfrak{G}(K_\mathfrak{P}/F_\mathfrak{p})$. Then $\varphi^{-1} \mathrm{Ad}\, u$ is defined over F and $\varphi^{-1} \mathrm{Ad}\, u(U'_{F_\mathfrak{p}}) = U_{F_\mathfrak{p}}$. Consequently, $U_{F_\mathfrak{p}}$ is one of the compact subgroups of the fourth paragraph.

To show that almost all $\mathfrak{p}$ are unramified, all we need do is observe that if π occurs in $L^2(G_F \setminus G_{\mathbb{Z}(F)})$, whatever the precise meaning of this is to be, and $\pi = \prod_\mathfrak{p} \otimes \pi_\mathfrak{p}$, then for almost all $\mathfrak{p}$, the restriction of $\pi_\mathfrak{p}$ to $U_{F_\mathfrak{p}}$ contains the trivial representation.

If $\mathfrak{p}$ is unramified let the homomorphism of $C_c(G_{F_\mathfrak{p}}, U_{F_\mathfrak{p}})$ associated to $\pi_\mathfrak{p}$ be $\chi_{t_\mathfrak{p}}$. To show that the product of the local L-function converges in a half plane it would be enough to show that there is a positive constant a such that for all unramified $\mathfrak{p}$ every eigenvalue of $\rho(t_\mathfrak{p} \sigma_{F_\mathfrak{p}})$ is bounded by $|\pi_\mathfrak{p}|^{-a}$. We may suppose that $\sigma_{F_\mathfrak{p}}(t_\mathfrak{p}) = t_\mathfrak{p}$. if $n = [K : F]$, then $(t_\mathfrak{p} \sigma_{F_\mathfrak{p}})^n = t_\mathfrak{p}^n$ so that we need only show that the eigenvalues of $\rho(t_\mathfrak{p})$ are bounded by $|\pi_\mathfrak{p}|^{-a}$. This we did in the previous paragraph.

7. Once the definitions are made we can begin to pose questions. My hope is that these questions have affirmative answers. The first question is the one initially posed.

Question 1. *Is it possible to define the local L-functions $L(s, \rho, \pi)$ and the local factors $\varepsilon(s, \rho, \pi, \psi_F)$ at the ramified primes so that if F is a global field $\pi = \prod \otimes \pi_\mathfrak{p}$, and*

$$L(s, \rho, \pi) = \prod_\mathfrak{p} L(s, \rho_\mathfrak{p}, \pi_\mathfrak{p})$$

then $L(s, \rho, \pi)$ is meromorphic in the entire complex plane with only a finite number of poles and satisfies the functional equation

$$L(s, \rho, \pi) = \varepsilon(s, \rho, \pi) L(1 - s, \tilde{\rho}, \pi)$$

and

$$\varepsilon(s,\rho,\pi)=\prod_{\mathfrak{p}}\varepsilon\left(s,\rho_{\mathfrak{p}},\pi_{\mathfrak{p}},\psi_{F_{\mathfrak{p}}}\right)\ .$$

The theory of Eisenstein series can be used [9] to give some novel instances in which this question has, in part, an affirmative answer. However, that theory does not suggest any method of attacking the general problem. If $G = GL(n)$ then $\hat{G}_F = GL(n,\mathbb{C})$. The work of Godement and earlier writers allows one to hope that the methods of Hecke and Tate can, once the representation theory of the general linear group over a local field is understood, be used to answer the first question when $G = GL(n)$ and ρ is the standard representation of $GL(n,\mathbb{C})$. The idea which led Artin to the general reciprocity law suggests that we try to answer it in general by answering a further series of questions. For the sake of precision, but not clarity, I write them down in an order opposite to that in which they suggest themselves. If G is defined over the local field F let $\Omega(G_F)$ be the set of equivalence classes of irreducible unitary representations of G_F.

Question 2. *Suppose G and G' are defined over the local field F, G is quasi-split and G' is obtained from G by an inner twisting. Then $\hat{G}_F = \hat{G}'_F$. Is there a correspondence F whose domain is $\Omega(G'_F)$ and whose range is contained in $\Omega(G_F)$ such that if $\pi = R(\pi')$ then $L(s,\rho,\pi) = L(s,\rho,\pi')$ for every representation ρ of $\hat{G}_F$?*

Notice that R is not required to be a function. I do not know whether or not to expect that

$$\varepsilon(s,\rho,\pi,\psi_F)=\varepsilon(s,\rho,\pi',\psi_F)\ .$$

One should, but I have not yet done so, look carefully at this question when F is the field of real numbers. For this one will of course need the work of Harish-Chandra.

Supposing that the second question has an affirmative answer, one can formulate a global version.

Question 3.* *Suppose that G and G' are defined over the global field F, G is quasi-split, and G' is obtained from G by an inner twisting. Suppose $\pi' = \prod_{\mathfrak{p}} \otimes \pi'_{\mathfrak{p}}$ occurs in $L^2(G'_F \setminus G'_{\mathbb{A}(F)})$. Choose for each $\mathfrak{p}$ a representation $\pi_{\mathfrak{p}}$ of $G_{F_{\mathfrak{p}}}$ such that $\pi_{\mathfrak{p}} = R(\pi'_{\mathfrak{p}})$. Does $\pi = \prod_{\mathfrak{p}} \otimes \pi_{\mathfrak{p}}$ occur in $L^2(G_F \backslash G_{\mathbb{A}(F)})$?*

Affirmative evidence is contained in papers of Eichler [3] and Shimizu [16] when $G = GL(2)$ and G' is the group of invertible elements in a quaternion algebra. Jacquet [16], whose work is not yet complete, is obtaining very general results for these groups.

* The question, in this crude form, does not always have an affirmative answer (cf. [6]). The proper question is certainly more subtle but not basically different.

Question 4. *Suppose G and G' are two quasi-split groups over the local field F. Let G split over K and let G' split over K' with $K \subseteq K'$. Let ψ be the natural map $\mathfrak{G}(K'/F) \to \mathfrak{G}(K/F)$. Suppse φ is a complex analytic homomorphism from $\hat{G}'_{K'/F}$ to $\hat{G}_{K/F}$ which makes*

$$\begin{array}{ccc} \hat{G}'_{K'/F} & \longrightarrow & \mathfrak{G}(K'/F) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\psi} \\ \hat{G}_{K/F} & \longrightarrow & \mathfrak{G}(K/F) \end{array}$$

commutative. Is there a correspondence R_φ with domain $\Omega(G'_F)$ whose range is contained in $\Omega(G_F)$ such that if $\pi = R_\varphi \pi'$, then, for every representation ρ of $\hat{G}_F$ and every non-trivial additive character ψ_F, $L(s,\rho,\pi) = L(s,\rho\circ\varphi,\pi')$ and $\varepsilon(s,\rho,\pi,\psi_F) = \varepsilon(s,\rho\circ\varphi,\pi',\psi_F)$?

R_φ should of course be functorial and, in an unramified situation, if π' is associated to the conjugacy class $t' \times \sigma'_F$, then π should be associated to $\varphi(t' \times \sigma'_F)$. I have not yet had a chance to look carefully at this question when F is the field of real numbers.

The question has a global form.

Question 5. *Suppose G and G' are two quasi-split groups over the global field F. Let G split over K and let G' split over K' with $K \subseteq K'$. Suppose φ is a complex analytic homomorphism from $\hat{G}'_{K'/F}$ to $\hat{G}_{K/F}$ which makes*

$$\begin{array}{ccc} \hat{G}'_{K'/F} & \longrightarrow & \mathfrak{G}(K'/F) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow \\ \hat{G}_{K/F} & \longrightarrow & \mathfrak{G}(K/F) \end{array}$$

commutative. If $\mathfrak{P}'$ is a prime of K', let $\mathfrak{P} = \mathfrak{P}' \cap K$ and let $\mathfrak{p} = \mathfrak{P}' \cap F$. φ determines a homomorphism $\varphi_\mathfrak{p}\colon \hat{G}'_{K'_{\mathfrak{P}'}/F_\mathfrak{p}} \to \hat{G}_{K_\mathfrak{P}/F_\mathfrak{p}}$ which makes

$$\begin{array}{ccc} \hat{G}'_{K'_{\mathfrak{P}'}/F_\mathfrak{p}} & \longrightarrow & \mathfrak{G}(K'_{\mathfrak{P}'}/F_\mathfrak{p}) \\ \Big\downarrow & & \Big\downarrow \\ \hat{G}_{K_\mathfrak{P}/F_\mathfrak{p}} & \longrightarrow & \mathfrak{G}(K_\mathfrak{P}/F_\mathfrak{p}) \end{array}$$

commutative. If $\pi' = \prod \otimes \pi'_\mathfrak{p}$ occurs in $L^2(G'_F \setminus F'_{\mathbb{A}(F)})$ choose for each $\mathfrak{p}$ a $\pi_\mathfrak{p} = R_{\varphi_\mathfrak{p}}(\pi'_\mathfrak{p})$. If $\pi = \prod\limits_{\mathfrak{p}} \otimes \pi_\mathfrak{p}$ does π occur in $L^2(G_F \setminus G_{\mathbb{A}(F)})$?

An affirmative answer to the third and fifth questions would allow us to solve the first question by examining automorphic forms on the general linear groups.

It is probably worthwhile to point out the difficulty of the fifth question by giving some examples. Take $G' = \{1\}$ $G = GL(1)$, K' any Galois extension of F and $K = F$. The assertion that, in this case, the last two questions have affirmative answers is the Artin reciprocity law.

Suppose G is quasi-split and $G' = T$. We may identify $\hat{G}'_F$ with $\hat{T} \times \mathfrak{G}(K/F)$ which is contained in $\hat{G}_F$. Thus we take $K' = K$. Let φ be the imbedding. In this case π' is a character of $G'_F \setminus G'_{\mathbb{A}(F)}$. The fourth question is, with certain reservations, answered affirmatively by the theory of induced representations. The fifth question is, with similar reservations, answered by the theory of Eisenstein series. The reservations are not important. I only want to point out that the theory of Eisenstein series is a prerequisite to the solution of these problems. with G as before, take $G'' = \{1\}$ and $K'' = K$ so that $\hat{G}''_F = \mathfrak{G}(K/F)$. Let ψ take σ in $\mathfrak{G}(K/F)$ to σ to $\hat{G}_F$. There is only one choice of π''. The associated space of automorphic forms on $G_F \setminus F_{\mathbb{A}(F)}$ should be the space of automorphic forms associated to the trivial character of $G'_F \setminus G'_{\mathbb{A}(F)}$. For this character all the reservations apply. I point out that the space associated to π'' is not the obvious one. It is not the space of constant functions. To prove its existence will require the theory of Eisenstein series.

Take $G = GL(2)$ and let G' be the multiplicative group of a separable quadratic extension K' of F. Take $K = F$. Then $\hat{G}'_F$ is a semi-direct product $(\mathbb{C}^* \times \mathbb{C}^*) \times \mathfrak{G}(K'/F)$. If σ is the non-trivial element of $\mathfrak{G}(K'/F)$, then $\sigma((t_1, t_2)) = (t_2, t_1)$. Let φ be defined by

$$\varphi\colon\ (t_1, t_2) \longrightarrow \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

$$\varphi\colon\ \sigma \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The existence of R_φ in the local case is a known fact (see for example [6]) in the theory of representations of $GL(2, F)$. An affirmative answer to the fifth question can be given by means of the Hecke theory [6] and by other means [15].

Let E be a separable extension of F and let G be the group over F obtained from $GL(2)$ over D by restriction of scalars. Let G' be $GL(2)$ over F and let $K' = K$ be any Galois extension containing E. Let X be the homogeneous space $\mathfrak{G}(K/E) \setminus \mathfrak{G}(K/F)$. Then $\hat{G}_F$ is the semi-direct product of $\prod_{x \in X} GL(2, \mathbb{C})$ and $\mathfrak{G}(K/F)$. If σ belongs to $\mathfrak{G}(K/F)$, then

$$\sigma\Big(\prod_{x \in X} A_x\Big)\sigma^{-1} = \prod_{x \in X} B_x$$

with $B_x = A_{x\sigma}$. Define φ by

$$\varphi(A \times \sigma)\Big(\prod_{x \in X} A\Big) \times \sigma \ .$$

Although not much is known about the fifth question in this case, the paper [2] of Doi and Naganuma is encouraging.

Suppose G and K are given. Let $G' = \{1\}$ and let K' be any Galois extension of F containing K. If F is a local field, the fourth question asks that, to every homomorphism φ of $\mathfrak{G}(K'/F)$ into $\hat{G}_F$ which makes

$$\begin{array}{ccc} \mathfrak{G}(K'/F) & \xrightarrow{\varphi} & \hat{G}_F \\ & \searrow & \downarrow \\ & & \mathfrak{G}(K/F) \end{array}$$

commutative, there be associated at least one irreducible unitary representation of G_F. If F is global, the fifth question asks that to φ there be associated a presentation of $G_{\mathbb{A}(F)}$ occurring in $L^2(G_F \backslash G_{\mathbb{A}(F)})$.

The L-functions we have introduced have been so defined that they include the Artin L-functions. However, Weil [17] has generalized the notion of an Artin L-function. The preceding observations suggest a relation between the generalized Artin L-function and the L-functions of this paper. Weil's definition requires the introduction of some locally compact groups – the Weil groups. If F is a local field, let C_F be the multiplicative group of F. If F is a global field, let C_F be the idèle class group. If K is a Galois extension of F, the Weil group $W_{K/F}$ is an extension

$$1 \to C_K \to W_{K/F} \to \mathfrak{G}(K/F) \to 1$$

of $\mathfrak{G}(K/F)$ by C_K. There is a canonical homomorphism $\tau_{K/F}$ of $W_{K/F}$ onto C_F. If F is a global field, $\mathfrak{P}$ a prime of K, and $\mathfrak{p} = F \cap \mathfrak{P}$, there is a homomorphism $\alpha_{\mathfrak{p}}$: $W_{K_{\mathfrak{P}}/F_{\mathfrak{p}}} \to W_{K/F}$. $\alpha_{\mathfrak{p}}$ is determined up to an inner automorphism. If σ is a representation of $W_{K/F}$, the class of $\sigma_{\mathfrak{p}} = \sigma \circ \alpha_{\mathfrak{p}}$ is independent of $\alpha_{\mathfrak{p}}$. By a representation σ of $W_{K/F}$ we understand a finite dimensional complex representation such that $\sigma(w)$ is semi-simple for all w in $W_{K/F}$.

If F is a local field and ψ_F a non-trivial additive character of F, then for any representation σ of $W_{K/F}$ we can define (cf. [11]) a local L-function $L(s,\sigma)$ and a factor $\varepsilon(s,\sigma,\psi_F)$. If F is a global field and σ is a representation of $W_{K/F}$, the associated L-function is

$$L(s,\sigma) = \prod_{\mathfrak{p}} L(s,\sigma_{\mathfrak{p}}) \ .$$

The product is taken over all primes, including the archimedean ones. If ψ_F is a non-trivial character of $F \setminus \mathbb{A}(F)$, then $\varepsilon(s, \sigma_\mathfrak{p}, \psi_{F_\mathfrak{p}})$ is 1 for almost all $\mathfrak{p}$,

$$\varepsilon(s, \sigma) = \prod_{\mathfrak{p}} \varepsilon(s, \sigma_\mathfrak{p}, \psi_{F_\mathfrak{p}})$$

is independent of ψ_F, and

$$L(s, \sigma) = \varepsilon(s, \sigma) L(1 - s, \tilde{\sigma})$$

if $\tilde{\sigma}$ is contragredient to σ.

Question 6. *Suppose G is quasi-split over the local field F and splits over the Galois extension K. Let $\hat{U}_F$ be a maximal compact subgroup of $\hat{G}_F$. Let K' be a Galois extension of F which contains K and let φ be a homomorphism of $W_{K'/F}$ onto $\hat{U}_F$ which makes*

$$\begin{array}{ccc} W_{K'/F} & \longrightarrow & \mathfrak{G}(K'/F) \\ \big\downarrow{\scriptstyle \varphi} & & \big\downarrow \\ \hat{U}_F & \longrightarrow & \mathfrak{G}(K/F) \end{array}$$

commutative. Is there an irreducible unitary representation $\pi(\varphi)$ of G_F such that, for every representation σ of $\hat{G}_F$, $L(s, \sigma, \pi(\varphi)) = L(s, \sigma \circ \varphi)$ and $\varepsilon(s, \sigma, \pi(\varphi), \psi_F) = \varepsilon(s, \sigma \circ \varphi, \psi_F)$?

Changing φ by an inner automorphism $\hat{U}_F$ will not change $\pi(\varphi)$, or at least not its equivalence class. If F is non-archimedean and K'/F is unramified, the composition of v, the valuation on F, and $\tau_{K'/F}$ defines a homomorphism ω of $W_{K/F}$ onto $\mathbb{Z}$. If $u = t \times \sigma_F$ belongs to $\hat{U}_F$, we could define φ by

$$\varphi(w) = u^{\omega(w)} \ .$$

Then $\pi(\varphi)$ would be the representation associated to the homomorphism χ_t of the Hecke algebra into $\mathbb{C}$.

We can also ask the question globally.

Question 7. *Suppose G is quasi-split over the global field F and splits over K. Let K' be a Galois extension of F containing K and let φ be a homomorphism of $W_{K'/F}$ into $\hat{U}_F$ which makes*

$$\begin{array}{ccc} W_{K'/F} & \longrightarrow & \mathfrak{G}(K'/F) \\ \big\downarrow{\scriptstyle \varphi} & & \big\downarrow \\ \hat{U}_F & \longrightarrow & \mathfrak{G}(K/F) \end{array}$$

commutative. If $\mathfrak{P}'$ is a prime of K' and $\mathfrak{p} = \mathfrak{P}' \cap F$, then $\varphi_\mathfrak{p} = \varphi \circ \alpha_\mathfrak{p}$ takes $W_{K'_{\mathfrak{P}'}/F_\mathfrak{p}}$ into $\hat{U}_{F_\mathfrak{p}}$. If $\pi(\varphi) = \prod_\mathfrak{p} \otimes \pi(\varphi_\mathfrak{p})$, does $\pi(\varphi)$ occur in $L^2(G_F \setminus G_{\mathbb{A}(F)})$?

Both questions have affirmative answers if G is abelian [10] and the correspondence $\varphi \to \pi(\varphi)$ is surjective. In this case our L-functions are all generalized Artin L-functions. If $G = GL(2)$ and $K = F$, it appears that the Hecke theory can be used to give an affirmative answer to both questions if it is assumed that certain of the generalized Artin L-functions have the expected analytic properties. If all goes well, the details will appear in [6].

I would like very much to end this series of questions with some reasonably precise questions about the relation of the L-functions of this paper to those associated to non-singular algebraic varieties. Unfortunately, I am not competent to do so. Since it may be of interest, I would like to ask one question about the L-functions associated to elliptic curves. If C is defined over a local field F of characteristic zero, I am going to associate to it a representation $\pi(C/F)$ of $GL(2, F)$. If C is defined over a global field F which is also characteristic zero, then for each prime $\mathfrak{p}$, $\pi(C/F_\mathfrak{p})$ is defined. Does $\pi = \prod_\mathfrak{p} \otimes \pi(C/F_\mathfrak{p})$ occur in $L^2(GL(2, F) \setminus GL(2, \mathbb{A}(F))$? If so, $L(s, \sigma, \pi)$, with σ the standard representation of $GL(2, \mathbb{C})$, whose analytic properties are known [6] will be one of the L-functions associated to the elliptic curve. There are examples on which the question can be tested. I hope to comment on them in [6].

To define $\pi(C/F)$, I use the result of Serre [14]. Suppose that F is non-archimedean and the j-invariant of C is integral. Take any prime ℓ different from the characteristic of the residue field and consider the ℓ-adic representation. There is a finite Galois extension K of F such that, if A is the maximal unramified extension of K, the ℓ-adic representation can be regarded as a representation of $\mathfrak{G}(A/F)$. There is a homomorphism of $W_{K/F}$ into $\mathfrak{G}(A/F)$. The ℓ-adic representation of $\mathfrak{G}(A/F)$ determines a representation φ of $W_{K/F}$ in $GL(2, R)$, where R is a finitely generated subfield of the ℓ-adic field $\mathbb{Q}_\ell$. Let σ be an isomorphism of R with a subfield fo $\mathbb{C}$. Then

$$\psi\colon\ w \to |\tau_{K/F}(w)|^{1/2} \varphi^\sigma(w)$$

is a representation of $W_{K/F}$ in a maximal compact subgroup of $GL(2, \mathbb{C})$. Let $\pi(C/F)$ be the representation $\pi(\psi)$ of Question 6. If C has good reduction, the class of ψ is independent of ℓ and σ. I do not know if this is so in general. It does not matter, because we do not demand that $\pi(C/F)$ be uniquely determined by C.

If the j-invariant is not integral, the ℓ-adic representation can be put in the form

$$\sigma \to \begin{pmatrix} \chi_1(\sigma) & * \\ 0 & \chi_2(\sigma) \end{pmatrix}$$

where χ_1 and χ_2 are two representations of the Galois group of the algebraic closure of F in the multiplicative group of $\mathbb{Q}_\ell$. If A is the maximal abelian extension of F, then χ_1 and χ_2 may be regarded as representations of $\mathfrak{G}(A/F)$. There is a canonical map of F^*, the multiplicative group of F, into $\mathfrak{G}(A/F)$. χ_1 and χ_2 thus define characters μ_1 and μ_2 of F^*. μ_1 and μ_2 take values in $\mathbb{Q}^*$ and $\mu_1\mu_2(x) = \mu_1\mu_2^{-1}(x) = |x|^{-1}$. In, for example, [6], there is associated to the pair of generalized characters $x \to |x|^{1/2}\mu_1(x)$ and $x \to |x|^{1/2}\mu_2(x)$ a unitary representation of $GL(2, F)$, a so-called special representation. This we take as $\pi(C/F)$.

If F is $\mathbb{C}$, take $\pi(C/F)$ to be the representation of $GL(2, \mathbb{C})$ associated to the map

$$s \to \begin{pmatrix} \frac{z}{|z|} & 0 \\ 0 & \frac{\bar{z}}{|z|} \end{pmatrix}$$

of $\mathbb{C}^* = W_{\mathbb{C}/\mathbb{C})}$ into $GL(2, \mathbb{C})$ by Question 6. $\mathbb{C}^*$ is of index two in $W_{\mathbb{C}/\mathbb{R}}$. The representation of $W_{\mathbb{C}/\mathbb{R}}$ induced from the character $z \to \frac{1}{|z|}$ of $\mathbb{C}^*$ has degree 2. If $F = \mathbb{R}$, let $\pi(C/F)$ be the representation of $GL(2, \mathbb{R})$ associated to the induced representation by Question 6.

8. I would like to finish up with some comments on the relation of the L-functions of this paper to Ramanujan's conjecture and its generalizations. Suppose $\pi = \prod \otimes \pi_{\mathfrak{p}}$ occurs in the space of cusp forms. The most general form of Ramanujan's conjecture would be that for all $\mathfrak{p}$ the character of $\pi_{\mathfrak{p}}$ is a tempered distribution [5]. However, neither the notion of a character nor that of a tempered distribution has been defined for non-archimedean fields. A weaker question is whether or not at all unramified non-archimedean primes the conjugacy class in $\hat{G}_F$ associated to $\pi_{\mathfrak{p}}$ meets $\hat{U}_F$ (cf. [13]). If this is so, it should be reflected in the behavior of the L-functions.

Suppose, to remove all ramification, that G is a Chevalley group and that $K = F = \mathbb{Q}$. Suppose also that each $\pi_{\mathfrak{p}}$ is unramified. If p is non-archimedean, there is associated to π_p a conjugacy class $\{t_p\}$ in $G_{\mathbb{Q}}$. We may take t_p in $\hat{T}$. The conjecture is that, for all λ in $\hat{L}$,

$$|\lambda(t_p)| = 1 \ .$$

Since there is no ramification at ∞, one can, as in [9], associate to π_∞ a semi-simple conjugacy class $\{X_\infty\}$ in the Lie algebra of $\hat{G}_{\mathbb{Q}}$. We may take X_∞ in the Lie algebra of $\hat{T}$. The conjecture at ∞ is that, for λ in $\hat{L}$,

$$Re\, \lambda(X_\infty) = 0 \ .$$

If σ is a complex analytic representation of $\hat{G}_{\mathbb{Q}}$, let $m(\lambda)$ be the multiplicity with which λ occurs in σ. Then

$$L(s,\sigma,\pi)=\prod_{\lambda}\left\{\pi\,\frac{-s(s+\lambda(\infty))}{2}\Gamma\left(\frac{2+\lambda(X_\infty)}{2}\right)\prod_p\frac{1}{1-\frac{\lambda(t_p)}{p^s}}\right\}^{m(\lambda)}\ .$$

If the conjecture is true, $L(s,\sigma,\pi)$ is analytic to the right of $Re\,s=1$ for all σ.

Let F be any non-archimedean local field and G any quasi-split group over F which splits over an unramified extension field. If f belongs to $C_c(G_F,U_F)$, let $f^*(g)=\bar{f}(g^{-1})$. If $\hat{f}$ and $\hat{f}^*$ are the images of f and f^* in $\Lambda^0(M)$, then $\hat{f}^*(\lambda)$ is the complex conjugate of $\hat{f}(-\lambda)$. If t belongs to $\hat{T}$, define t^* by the condition that $\lambda(t^*)=\overline{\lambda(t^{-1})}$ for all λ in $\hat{L}$. The complex conjugate of $\chi_t(t^*)$ is

$$\Sigma\hat{f}(-\lambda)\overline{\lambda(t)}=\Sigma\hat{f}(\lambda)\lambda(t^*)=\chi_{t^*}(f)\ .$$

If χ_t is the homomorphism associated to a unitary representation, then $\chi_t(f^*)$ is the complex conjugate of $\chi_t(t)$ for all f so that $t\times\sigma_F$ is conjugate to $t^*\times\sigma_F$ and for any representation ρ of $\hat{G}_F$, the complex conjugate of trace $\rho(t\times\sigma_F)$ is trace $\tilde{\rho}(t\times\sigma_F)$ if $\tilde{\rho}$ is the contragredient of ρ. In the case under consideration, when $K=F$ this means that trace $\rho(t_p)$ is the complex conjugate of trace $\tilde{\rho}(t_p)$. A similar argument can be applied at the infinite prime so show that the eigenvalues of $\rho(X_\infty)$ are the complex conjugates of the eigenvalues of $\tilde{\rho}(X_\infty)$.

Suppose $L(s,\sigma,\pi)$ is analytic to the right of $Re\,s=1$ for all σ. Since the Γ-function has no zeros,

$$\prod_{\lambda}\left\{\prod_p\frac{1}{1-\frac{\lambda(t_p)}{p^s}}\right\}^{m(\lambda)} \tag{C}$$

is also. Let σ be $\rho\otimes\tilde{\rho}$. Then the logarithm of this Dirichlet series is

$$\sum_p\sum_{n=1}^{\infty}\frac{1}{n}\frac{\text{trace }\sigma^n(t_p)}{p^{ns}}\ .$$

Since

$$\text{trace }\sigma^n(t_p)=\text{trace }\rho^n(t_p)\text{trace }\tilde{\rho}^n(t_p)=|\text{trace }\rho^n(t_p)|^2$$

the series for the logarithm has positive coefficients. Thus, the original series does too. By Landau's theorem, it converges absolutely for $Re\,s>1$ and so does the series for its logarithm. In particular,

$$\det\left(1-\frac{\sigma(t_p)}{p^s}\right)$$

does not vanish for $Re\, s > 1$ so that the eigenvalues of $\sigma(t_p)$ are all less than or equal to p in absolute value. If λ is a weight, choose ρ such that $m\lambda$ occurs in ρ. Then $(m\lambda)(t_p) = \lambda(t_p)^m$ is an eigenvalue of $\rho(t_p)$ and $\overline{\lambda(t_p)}^m$ is an eigenvalue of $\tilde{\rho}$, so that $|\lambda(t_p)|^{2m}$ is an eigenvalue of σ and

$$|\lambda(t_p)| \leq p^{\frac{1}{2m}}$$

for all m and all λ. Thus, $|\lambda(t_p)| \leq 1$ for all λ. Replacing λ by $-\lambda$, we see that $|\lambda(t_p)| = 1$ for all λ. Since the function defined by (C) cannot vanish for $Re\, s > 1$ when $\sigma = \rho \otimes \tilde{\rho}$, the function

$$\prod_{\lambda} \Gamma \left(\frac{2 + \lambda(X_\infty)}{2} \right)^{m(\lambda)}$$

must be analytic for $Re\, s > 1$. This implies that

$$Re\, \lambda(X_\infty) \geq -1$$

if $m(\lambda) > 0$. The same argument as before leads to the conclusion that $Re\, \lambda(X_\infty) = 0$ for all λ.

Granted the generalizations of Ramanujan's conjecture, one can ask about the asymptotic distribution of the conjugacy classes $\{t_p\}$. I can make no guesses about the answer. In general, it is not possible to compute the eigenvalues of the Hecke operators in an elementary fashion. Thus, Question 7 cannot be expected to lead by itself to elementary reciprocity laws. However, when the groups $G_{F_\mathfrak{p}}$ at the infinite primes are abelian or compact, these eigenvalues should have an elementary meaning. Thus, Question 7, together with some information on the range of the correspondences of Question 3, may eventually lead to elementary, but extremely complicated, reciprocity laws. At the present it is impossible even to speculate.

References

1. F. Bruhat and J. Tits, *Groupes algébriques simples sur un corps local*, Driebergen Conference on Local Fields, Springer-Verlag, 1967.
2. K. Doi and H. Naganuma, *On the algebraic curves uniformized by arithmetical automorphic functions*, Ann. of Math. vol. 86 (1967).
3. M. Eichler, *Quadratische Formen und Modulfunktion*, Acta Arith. vol. 4 (1958).
4. F. Gantmacher, *Canonical representation of automorphisms of a complex semi-simple Lie group*, Mat. Sb. vol. 47 (1939).
5. Harish-Chandra, *Discrete Series for Semi-Simple Lie Groups* II, Acta Math. vol. 116 (1966).
6. H. Jacquet and R. P. Langlands, *Automorphic Forms on $GL(2)$*, in preparation.
7. N. Jacobson, *Lie Algebras*, Interscience, 1962.
8. B. Kostant, *Lie Algebraic Cohomology and the Generalized Borel-Weil Theorem*, Ann. of Math. vol. 74 (1961).
9. R. P. Langlands, *Euler Products*, Lecture Notes, Yale University (1967).
10. ________________, *Representations of Abelian Algebraic Groups*, Notes, Yale University (1968).
11. ________________, *On the Functional Equation of the Artin L-functions*, in preparation.
12. I. Satake, *Theory of Spherical Functions on Reductive Algebraic Groups over $\mathfrak{p}$-adic Fields*, Publ. Math. No. 18, I.H.E.S (1963).
13. _________, *Spherical Functions and Ramanujan Conjecture*, in *Algebraic Groups and Discontinuous Subgroups*, Amer. Math. Soc. (1966).
14. J. P. Serre, *Groupes de Lie ℓ-adic attachés aux courbes elliptiques*, Colloque de Clermont-Ferrand (1964).
15. J. A. Shalika and S. Tanaka, *On an Explicit Construction of a Certain Class of Automorphic Forms*, Notes, Institute for Advanced Study (1968).
16. H. Shimizu, *On Zeta functions of quaternion algebras*, Ann. of Math. vol. 81 (1966).
17. A. Weil, *Sur la Theorie du Corps de Classes*, Jour. Math. Soc. Japan vol. 3 (1951).