Mathematics 256 — questions and answers

Question [cooling.4] Solve the differential equation & initial conditions

$$y' + y = t, \quad y(0) = 0 \ .$$

Answer [cooling.4] From the formula,

$$y = e^{-t} \int_0^t s e^s \, ds \ .$$

Now we have

$$\int x e^{cx} \, dx = \frac{x e^{cx}}{c} - \frac{e^{cx}}{c^2}$$

and therefore

$$\int_0^t s e^s \, ds = [s e^s - e^s]_0^t = t e^t - e^t + 1 \ .$$

We can check this result by substituting back into the differential equation.

Question [cooling.6] Suppose we have two radioactive substances A and B. The substance A decays into B with a relaxation time of τ_A, and B decays into the stable substance C with relaxation time τ_B. Let y_A, y_B, y_C be the amounts of each substance (depending on time). Write down three first order differential equations describing the relationship between the three quantities.

Answer [cooling.6]

$$\begin{aligned} y_A' &= -\frac{y_A}{\tau_A} \\ y_B' &= \frac{y_A}{\tau_A} - \frac{y_B}{\tau_B} \\ y_C' &= \frac{y_B}{\tau_B} \end{aligned}$$

Question [cooling.6a] (Continuing) Check your answer by verifying that the total amount of all three doesn't change.

Answer [cooling.6a] Add the equations together to see that $y_A' + y_B' + y_C' = 0$.

Question [cooling.7] (Continuing) Assuming that we start with an amount y_0 of substance A at $t = 0$ and no other substances, find an expression for y_A and y_B at time t.

Answer [cooling.7] We get $y_A = y_0 \, e^{-t/\tau_A}$ immediately since the equation for y_A depends only on y_A. The equation for y_B is then

$$\begin{aligned} y_B' &= \frac{y_0 \, e^{-t/\tau_A}}{\tau_A} - \frac{y_B}{\tau_B} \\ &= \frac{y_0}{\tau_A} \, e^{-t/\tau_A} - \frac{y_B}{\tau_B} \end{aligned}$$

This has a long-term solution

$$y_B = C e^{-t/\tau_A}$$

where we can find out what C is by substituting into the differential equation.

$$\begin{aligned} y_B &= Ce^{-t/\tau_A} \\ y_B' &= -(C/\tau_A)e^{-t/\tau_A} \\ y_B' + \frac{y_B}{\tau_B} &= -\frac{C}{\tau_A} + \frac{C}{\tau_B} \\ &= \frac{y_0}{\tau_B} \\ C\left(-\frac{1}{\tau_A} + \frac{1}{\tau_B}\right) &= \frac{y_0}{\tau_A} \\ C &= \frac{y_0 \tau_B}{\tau_A - \tau_B} \ . \end{aligned}$$

and then since $y_B(0) = 0$ the full solution

$$C(e^{-t/\tau_A} - e^{-t/\tau_B}) \ .$$

Question [cooling.10] Find a formula for general solution of

$$y' = a(t)y \ .$$

Answer [cooling.10] We separate:

$$\begin{aligned} \frac{dy}{y} &= a(t)\, dt \\ \log y &= \int a(t)\, dt \\ y = Ce^{A(t)} \end{aligned}$$

if $A(t)$ is any indefinite integral of $a(t)$, and $C = e^c$ for an integration constant c.

Question [cooling.11] By mimicking the derivation for the solution of $y' = ay + b(t)$, find a formula for the general solution of

$$y' = a(t)y + b(t) \ .$$

Answer [cooling.11] We have

$$\begin{aligned} y' - a(t)y &= b(t) \\ e^{A(t)}y' - A'(t)e^{A(t)}y &= e^{A(t)}b(t) \\ \frac{d}{dt}ye^{A(t)} &= e^{A(t)}b(t) \\ ye^{A(t)} &= \int e^{A(t)}b(t)\, dt \\ y &= e^{-A(t)} \int^t e^{A(s)}b(s)\, ds \end{aligned}$$

if

$$A(t) = \int^t a(s)\, ds \ .$$

Question [cplx.1] Find $1/(1+i)$, $1/(3+2i)$.

Answer [cplx.1]

$$\frac{1-i}{2}, \quad \frac{3-2i}{13}\ .$$

Question [cplx.2] Write down i^n for $n = -4$ to $n = 8$. For $n = 101$.

Answer [cplx.2]

$$i^n = \begin{cases} 1 & n = -4,\, 0,\, 4,\, \ldots \\ i & n = -3,\, 1,\, 5,\, \ldots \\ -1 & n = -2,\, 2,\, 6,\, \ldots \\ -i & n = -1,\, 3,\, 7,\, \ldots \end{cases}$$

Question [cplx.4] By writing $(\cos\theta + i\sin\theta)^3$ in two ways, find a formula for $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$. (Hint: First expand $(a+b)^3$.)

Answer [cplx.4]

$$\begin{aligned} (\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\sin\theta + 3i^2\cos\theta\sin^2\theta + \sin^3\theta \\ &= \cos 3\theta + \sin 3\theta \\ \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \end{aligned}$$

Question [cplx.5] (a) Find and plot in the (x, y) plane all the roots of $z^3 = 1$. (b) Of $z^4 = 1$. (c) Of $z^8 = 1$. (d) Of $z^4 = 2$.

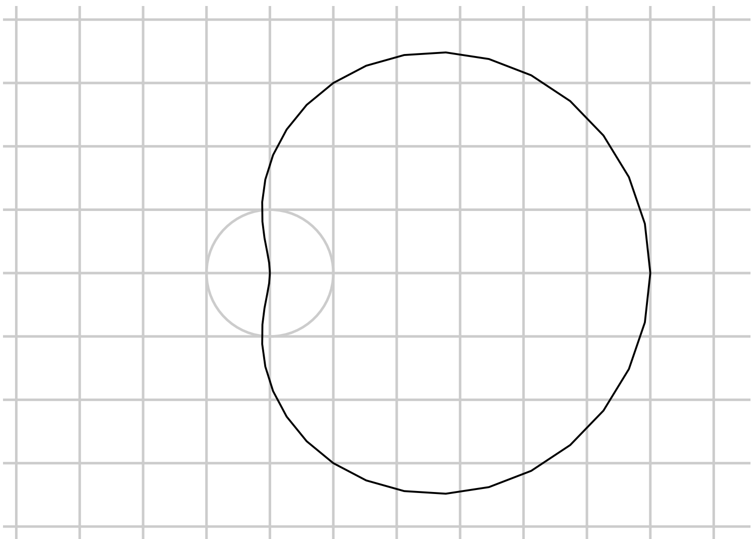
Answer [cplx.5] (a) 1, $\cos(2\pi/3) \pm i\sin(2\pi/3)$; (b) ± 1, $\pm i$; (c) ± 1, $\pm i$, $\pm 1/\sqrt{2} \pm i/\sqrt{2}$; (d) $\pm\sqrt[4]{2}$, $\pm i\sqrt[4]{2}$.

Question [cplx.6] Plot roughly the path traversed by

$$z^2 - 3z + 2$$

as z moves around the circle $\|z\| = 1$.

Answer [cplx.6]

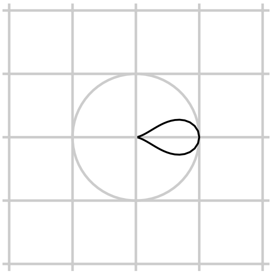


Question [cplx.7] Plot roughly the path traversed by the complex numbers

$$\frac{1}{1 + z - z^2}$$

as z goes from $-i\infty$ to $i\infty$ along the imaginary axis. (Hint: do 0 to $i\infty$ first.)

Answer [cplx.7]



Question [cplx.10] Find the integral

$$\int te^{ct}\, dt$$

by parts. Find the integral

$$\int t \cos t\, dt$$

by applying this for a suitable complex number c.

Answer [cplx.10]

$$\int te^{ct}\, dt = \frac{te^{ct}}{c} - \frac{e^{ct}}{c^2}\,.$$

Set $c = i$. What we are loking for is the real part of

$$= \frac{te^{it}}{i} - \frac{e^{it}}{i^2}$$

which is $t \sin t + \cos t$.

Question [eulers.2] If we use Euler's method to solve

$$y' = -y^2 + t, \quad y(0) = 0.5$$

we get the following estimates:

Δt	$y(1)$
0.100000	0.682953
0.050000	0.693979
0.025000	0.699202

Approximately how many steps would it take to estimate $y(1)$ correctly to 6 decimals, using Euler's method?

Answer [eulers.2] These data suggest that the error is about $0.699202 - 0.693979 = 0.005$ for step size 0.025. For an error of 0.0000001 we would use a step size of about $(0.000001/0.005) \cdot 0.025 = 0.000005$, or $200,000$ steps.

Question [falling.2] At $t = 1265$, $r = 6370.408$, while at $t = 1266$ it is at $r = 6363.676$. Calculate the moment of contact to within 0.1 second.

Answer [falling.2] At $t = 1265$, $r = 6370.408$, while at $t = 1266$ it is at $r = 6363.676$. In this short interval, the function $r(t)$ is essentially linear. Hence the moment of contact is

$$1265 + (0.408)/(6370.408 - 6363.676) = 1265.06\,.$$

Question [falling.3] Suppose that gravitational acceleration at $10,000$ km were the same as it is at the Earth's radius. How long would it take our object to fall to the surface?

Answer [falling.3] $t = \sqrt{2(10000 - 6370)/0.0098} = 861$ seconds.

Question [falling.4] Suppose that the acceleration during all of the fall were what it is at $10,000$ km. How long would the fall take?

Answer [falling.4] $t = \sqrt{2(10000 - 6370)/0.00398} = 1351$ seconds.

Question [falling.5] Suppose that near the Earth's surface an object is thrown down from height h_0 with speed v_0. What height is it at time t? How long to fall a distance Δh?

Answer [falling.5] $h = h_0 - v_0 t - gt^2/2$, $\Delta h = v_0\,\Delta t + g(\Delta t)^2/2$, solve for Δt:

$$\Delta t = \frac{-v_0 \pm \sqrt{v_0^2 + 2g\,\Delta h}}{g} \ .$$

We must choose the positive sign because $\Delta t > 0$.

Question [falling.6] The differential equation in this section is rather special, since energy is conserved as the object falls. The kinetic energy is $mv^2/2$ and the potential energy is $-mk/r$ (a negative coefficient because the object loses potential energy as it falls). Recall that potential energy is defined up to some fixed constant, and here that constant is chosen to make the potential energy vanish at $r = \infty$. (a) Write down the expression for total energy. (b) Differentiate it with respect to time, and use the second order differential equation we derived above for r, to show that energy is in fact constant. (c) Suppose the object falls from radius r_0 with no initial speed. What is its initial energy? (d) Solve the equation expressing conservation of energy to find an expression for r' at any radius r.

Answer [falling.6] (a) The energy is $mv^2 - mk/r$. (b) Differentiating we get

$$mvv' + mkr'/r^2 = mr'r'' + mkr'/r^2 = 0 \ .$$

(c) and (d) If initial height is r_0 we get

$$mv^2/2 - mk/r = -mk/r_0, \quad r' = v = \sqrt{2k}\sqrt{\frac{1}{r} - \frac{1}{r_0}} = \sqrt{2gR^2}\sqrt{\frac{1}{r} - \frac{1}{r_0}} \ .$$

Question [falling.7] How accurate is the table we calculated? If we run the calculations with smaller and smaller values of Δt we expect more and more accurate approximations. Here are some tables for a sample calculation:

$\Delta t = 1$:

900.0	8293.452	−4.0470	−0.00578

$\Delta = 0.5$:

900.0	8292.197	−4.0480	−0.00578

$\Delta = 0.25$:

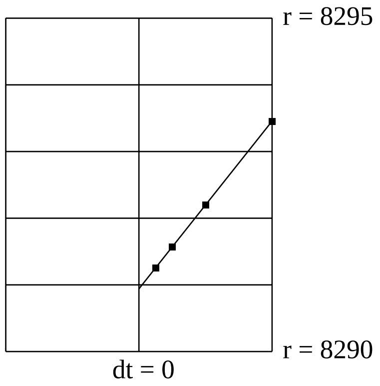
900.0	8291.569	−4.0485	−0.00578

$\Delta = 0.125$:

900.0	8291.255	−4.0487	−0.00578

Plot on a graph the calculated value of $r(900)$ versus the time interval Δt (four points altogether). Use the graph and a few calculations to estimate $r(900)$ as accurately as you can.

Answer [falling.7] Here is the plot:



The points seem to lie on a line, and we estimate $r(900) = 8290.94$.

Question [periodic.3] Write down the differential equation satisfied by a small object in a room with oscillating temperature $\theta_{\mathrm{env}}(t) = \cos t$, and a relaxation time of $\tau = 1$. Write down a formula for $\theta(t)$ if $\theta(0) = 100°$. Write down and then graph the transient and steady state components.

Answer [periodic.3]

$$\theta' = -\theta + \cos t) \ .$$

The solution is

$$100e^{-t} + e^{-t}\int_0^t e^s \cos s\, ds = \frac{1}{2}\big(\cos t + \sin t\big) + 99\ 1/2e^{-t} \ .$$

The steady state component is the first term, transient the second.

Question [periodic.5] Find a formula for the solution of Newton's cooling equation when the room temperature is t (a constantly rising room temperature), initial temperature θ_0. What is $\theta(t)$?

Answer [periodic.5]

$$\theta = \tau e^{-t/\tau} + (t - \tau)$$

which tends to $t - \tau$ as t becomes large, a simple lag.

Question [periodic.7] A cup of coffee initially at 100° cools to 40° after 10 minutes in a 20° room, and is then put in an oven at 120° for 5 minutes. What is its temperature then?

Answer [periodic.7] We can find τ be solving $3/4 = e^{-10/\tau}$, getting $\tau = 10/\log_e(4/3)$. After 5 minutes in the oven, it has temperature

$$120 - 80e^{-5/\tau} = 120 - 80 \cdot (3/4)^{1/2} \ .$$

Question [periodic.8] Find the general solution of

$$y' = -y + 1 + e^{-t} + \cos t \ .$$

Answer [periodic.8] We just solve each piece separately, and get

$$Ce^{-t} + 1 + te^{-t} + \frac{1}{2}(\cos t + \sin t) \ .$$

Question [periodic.9] Suppose we are looking at a cup of coffee initially at 100° where the relaxation time is 20 minutes, in an environment whose temperature is fluctuating at a simple frequency of period 10 minutes according to the formula

$$\theta_{\mathrm{env}}(t) = 5\cos 2\pi(t/10) \ .$$

What is the temperature of the coffee after 20 minutes?

Answer [periodic.9] Let $\omega = 2\pi/10$. The differential equation is

$$\theta' = -\frac{\theta - 5\cos\omega t}{20} \ .$$

We replace it by

$$\theta' + \frac{\theta}{20} = \frac{1}{4}e^{i\omega t}$$

with steady state solution $(C/4)e^{i\omega t}$,

$$C = \frac{1}{(1/20) + i\omega} \ .$$

The solution is the real part of

$$\theta(0)e^{-t/20} + \frac{1}{4}\frac{1}{(1/20) + i\omega}(e^{i\omega t} - e^{-t/20})$$

and at $t = 20$ we get

$$\theta_0 e^{-1} + \frac{1}{4}\frac{1}{(1/20) + i\omega}(1 - e^{-1}) \ldots$$