Chapter 10. Summary—time invariant physical systems of second order

Second order equations, even those with constant coefficients, are both complicated and important. A review will be helpful. A small number of new ideas will be added here, too.

1. All those different frequencies

There are several frequencies associated to second order differential equations in which the friction term is relatively light. The first is the response frequency ω_0 of the free system with no friction at all. The second is the quasi-frequency ω_{qf} which controls the rate of vibration of the decaying response to the associated homogeneous equation. The third is the resonant frequency, the frequency for which input causes the maximum amplitude of response.

If the equation is

$$my'' + cy' + ky = F(t)$$

where y is the displacement from equilibrium; m is mass; c is a damping coefficient, so that the friction term is proportional to velocity; k is a constant such that the force restoring the system to equilibrium is ky, then the formulas for these frequencies are

$$\begin{aligned}
\omega_0 &= \sqrt{k/m} \\
\omega_{\mathrm{qf}} &= \sqrt{4km - c^2}/2m \\
&= \sqrt{k/m - c^2/4km} \\
&= \omega_0\sqrt{1 - \gamma^2/4} \\
\omega_{\mathrm{res}} &= \sqrt{4km - c^2}/2m \\
&= \sqrt{k/m - c^2/2km} \\
&= \omega_0\sqrt{1 - \gamma^2/2}
\end{aligned}$$

where $\gamma = c/\sqrt{km}$ is the normalized friction. These formulas only make sense when the quantities under the square root signs are non-negative, which means that c is relatively small. In particular, the quasi-frequency makes no sense (there is no vibration for the solutions of the homogeneous equation) when $c^2 > 4km$ but resonance disappears in the larger range $c^2 > 2km$. When they all make sense, we have the relationship

$$\omega_{\mathrm{res}} \le \omega_{\mathrm{qf}} \le \omega_0 \ .$$

The three are the same only when there is no friction.

Second order differential equations like

$$my'' + cy' + ky = F(t)$$

with constant coefficients are probably the most important topic in the course.

2. Model physical systems

In most practical examples m, c, and k are all positive, but an important limiting case is when $c = 0$ as well. I shall assume in the rest of this note that $m > 0$, $k > 0$, $c \ge 0$.

The simplest physical models for this equation are (1) a weight of mass hanging on a spring obeying Hooke's law, and (2) an electric circuit consisting of a simple loop, with a resistor, capacitor, inductor, and voltage source all in a series. In the first case m is the mass of the weight, k the constant of proportionality in Hooke's law, c a friction coefficient, and $F(t)$ an external force driving the system up and down vertically. In the second example the constants are usually relabelled so the equation becomes either

$$LQ'' + RQ' + \frac{Q}{C} = E(t)$$

or

$$LI'' + RI' + \frac{I}{C} = E'(t)$$

where L is inductance, R resistance, C capacitance, $E(t)$ the voltage input, Q charge across the capacitor, and I the current in the circuit.

A system where $c = 0$ is one where energy conserved. The rate of energy loss in general in the system is proportional to $c(y')^2$.

The equation is linear, which means that the left hand side $L(y) = my'' + cy' + ky$ is a linear function of y. Linearity means that if y_1 is a solution of the equation

$$y'' + cy' + my = F_1(t)$$

and y_2 of the equation

$$y'' + cy' + ky = F_2(t)$$

then the linear combination $c_1y_1 + c_2y_2$ is a solution of

$$my'' + cy' + ky = c_1F_1(t) + c_2F_2(t) \ .$$

There is a formula for the solution of such an equation with an arbitrary input function $F(t)$, but the formula is not so useful. In practice there are a small number of cases which are important.

3. The homogeneous case

Here $F(t) = 0$. In this case the physical system being modelled is said to be free, and the equation

$$my'' + cy' + ky = 0$$

is called homogeneous. In this case we find all solutions by taking linear combinations of two particularly simple ones. There are three different cases, depending on the behaviour of the roots of the characteristic polynomial

$$m\lambda^2 + c\lambda + k = 0 \ .$$

The three cases are:

(1) The roots are λ_1 and λ_2, two distinct real numbers. The basic solutions are

$$e^{\lambda_1 t}, \quad e^{\lambda_2 t} \ .$$

Both roots are negative, and these solutions decay exponentially fast as $t \to \infty$. For almost all solutions both constants will be non-zero, and the rate of decay will be that of the term with the root of smallest absolute magnitude.

(2) The roots are two conjugate complex numbers $a \pm bi$. There are exponential solutions $e^{(a\pm bi)t}$ which give rise by Euler's formula to the basic solutions

$$e^{at}\cos bt, \quad e^{at}\sin bt \ .$$

Both are functions which oscillate and decay all at the same time. The frequency b of the oscillation is called the quasi-frequency of the equation.

(3) There is only one root λ. The basic solutions are

$$e^{\lambda t}, \quad te^{\lambda t} .$$

Both solutions decay exponentially, the second slightly less rapidly than the first.

4. Simple periodic input

Here $F(t)$ is a linear combination of $\cos \omega t$ and $\sin \omega t$, and may be expressed as $F_0 \cos(\omega t - \alpha)$, where F_0 is the magnitude of the input, ω its frequency, and α its phase. Explicitly, if

$$F(t) = A \cos \omega t + B \sin \omega t$$

Then

$$\begin{aligned} F_0 &= \sqrt{A^2 + B^2} \\ \cos \alpha &= A/F_0 \\ \sin \alpha &= B/F_0 \end{aligned}$$

Any solution to the differential equation is the sum of two parts. The first is a solution of the homogeneous equation. As $t \to \infty$ it decays exponentially as long as $c > 0$, and is then called the transient component. The time it takes to decay by a certain factor is called its relaxation time. The second part is of the same form as the input, except that its magnitude and phase are different. Explicitly it is

$$\frac{F_0}{R} \cos(\omega t - \alpha - \theta)$$

where

$$\begin{aligned} R &= \sqrt{(k - m\omega^2)^2 + c^2\omega^2} \\ \cos \theta &= \frac{k - m\omega^2}{R} \\ \sin \theta &= \frac{c\omega}{R} . \end{aligned}$$

This component of the solution is called the steady state component.

The general solution of the equation will involve two constants included in the transient component, usually determined by initial conditions. The effects of the initial conditions die out with time, and all solutions will be essentially the steady state solution.

In other words, the relation between the input $F(t)$ and the steady state solution—the output—is that the output is obtained from the input by an amplification and a shift in phase.

These formulas become somewhat simpler if we think of the input $F(t)$ as the real part of complex input $F_0 e^{i\omega t}$. Then the steady state component is

$$\frac{F_0 e^{i\omega t}}{m(i\omega)^2 + c(i\omega) + k} = \frac{F_0 e^{i\omega t}}{(k - m\omega^2) + ic\omega} .$$

In these terms R is the magnitude and θ the phase of the complex number in the denominator.

In most situations it is important to know how the amplification factor $1/R$ varies with ω. If c is small enough then there is a value of ω for which this amplification factor is relatively large. It is called the resonant frequency. If c is large, however, the amplification factor will be greatest at $\omega = 0$ and no resonance phenomenon occurs. If resonance does occur, then it is significant only when the resonant frequency lies close to $\omega_0 = \sqrt{k/m}$, the resonant frequency of a system where $c = 0$, and in fact friction is almost negligible.

So: small friction means (1) slowly oscillating exponentially decaying transient solutions and (2) a large amplification factor for input frequencies near the resonant frequency, while large friction means (1) rapidly decaying transient solutions and (2) very small steady state solution.

5. The exceptional case

The formula above makes sense unless the denominator

$$(k - m\omega^2) + ic\omega$$

is equal to 0. This can happen only if

$$\omega = \sqrt{k/m}, \quad c = 0$$

Here the physical model is one without friction, where the input is in synchronization with the frequency of the free system, and the magnitude of the output grows without limit. There is no steady state solution, since it is no longer true that initial effects die out. But as $t \to \infty$ all solutions to

$$my'' + ky = \cos \omega t$$

look like

$$\frac{t \sin \omega t}{2m\omega}$$

as $t \to \infty$, since the solution to the homogeneous equation at least remains bounded, and might be considered a relative transient.

Of course in realistic examples c is never exactly 0, and solutions never go to ∞, but this is a reasonable approximation for systems with very low friction and small initial time segments. It is also the case you want to avoid in engineering applications, since it will lead to disaster.

6. Step function input

The final important case is when $F(t)$ is equal to periodic input cut off outside a fixed interval of time. Fir example we could have

$$F(t) = \begin{cases} e^{i\omega t} & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Think of this as modelling a weight which is given a blow, or an electric circuit where the outside source is switched on and off. The best way to solve this is to solve the equation separately in each of the ranges $(-\infty, 0]$, $[0, T]$, $[T, \infty)$, and then match initial conditions at the boundary points $t = 0$, $t = T$.

7. An extended example

Let's solve

$$y'' + 2y' + 2y = \cos 3t, \quad y(0) = y'(0) = 0 \ .$$

The roots of the characteristic equation are $-1 \pm i$. The transient solutions are of the form

$$ae^{-t} \cos t + be^{-t} \sin t \ .$$

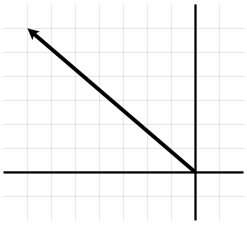
The steady state solution is the real part of

$$\frac{e^{3it}}{(3i)^2 + 2(3i) + 2} = \frac{e^{3it}}{-7 + 6i} = \frac{-7 - 6i}{-7 - 6i} \, \frac{\cos 3t + i \sin 3t}{-7 + 6i} = \frac{-7 \cos 3t + 6 \sin 3t}{85} \ .$$

We have

$$R = \sqrt{49 + 36} = \sqrt{85}$$

and θ equal to the angle of the vector in this picture:



which is to say about $2.433^r = 139°$.

Therefore the solution and its derivative are

$$y = ae^{-t}\cos t + be^{-t}\sin t + \frac{-7\cos 3t + 6\sin 3t}{85}$$
$$y' = -ae^{-t}\cos t - ae^{-t}\sin t - be^{-t}\sin t + be^{-t}\cos t + \frac{21\sin 3t + 18\cos 3t}{85}$$

We have

$$y(0) = a - \frac{7}{85}$$
$$y'(0) = -a + b + \frac{18}{85}\ .$$

so we get

$$a = \frac{7}{85}, \quad b = -\frac{11}{85}\ .$$

If the input is $e^{i\omega t}$ the denominator of the steady state solution is

$$(2-\omega^2) + 2i\omega$$

The square of its magnitude is

$$(2-\omega^2)^2 + 4\omega^2 = \text{say} X(\omega)$$

and to find the maximum point we set

$$X'(\omega) = 2(-2\omega)(2-\omega^2) + 8\omega = 0\ .$$

We get $\omega = 0$ as the only place where this occurs. There is therefore no resonant frequency.

Exercise 7.1. *For which values of c is there no resonant frequency for the equation*

$$y'' + cy' + 2y = 0?$$