Chapter 14. Systems of weights

In applying Newton's Law to a large physical structure, we must take the mutual dependence of its components into account. The differential equations that govern the motion in all circumstances will be extremely complicated, since how structures respond to stress will be very complicated. In particular, extreme circumstances will lead to grossly non-linear phenomena such as warping or over-heating. Nonetheless, at least near equilibrium we can approximate the equations by linear ones, and use matrix algebra to understand the response. This part of the course is in fact concerned almost exclusively with systems of **linear** differential equations.

The physical systems we spend time on in this course are relatively simple. It may seem that we are going to a lot of work for unrealistic examples. The point is that many quite distinct physical systems are modelled mathematically by the same type of differential equations. The main purpose of looking at the systems we do look at is to give the mathematics some intuitive content, not to explain realistic physics.

But before we begin the details, we shall look very briefly at a system at once more realistic and only slightly too complicated to examine more closely—a book shelf. Books sit on top of the shelf and bend it downwards. The first question you might ask is: *Why doesn't the shelf just fall in under the load?* The answer is that as the books push down upon the shelf the internal structure of the material acts as a network of small (very small!) springs being stretched. The springs on the top are perhaps compressed, those on the bottom stretched, and the net effect is to push back up against the load. The shelf stops bending when some sort of equilibrium is reached. Exactly what shape is reached leads to the first common question about physical systems:

- *What is the equilibrium position of the system?*

If for some reason the books are bounced up and down on the shelf (or you look instead at a person at the end of a diving board) you meet the second common question:

- *What happens to the system when it is perturbed slightly from equilibrium?*

Even a book shelf is quite complicated to analyze. Weights on a spring, which we look at next, make up the simplest example where the answers are non-trivial, and where typical phenomena occur.

1. Setting up

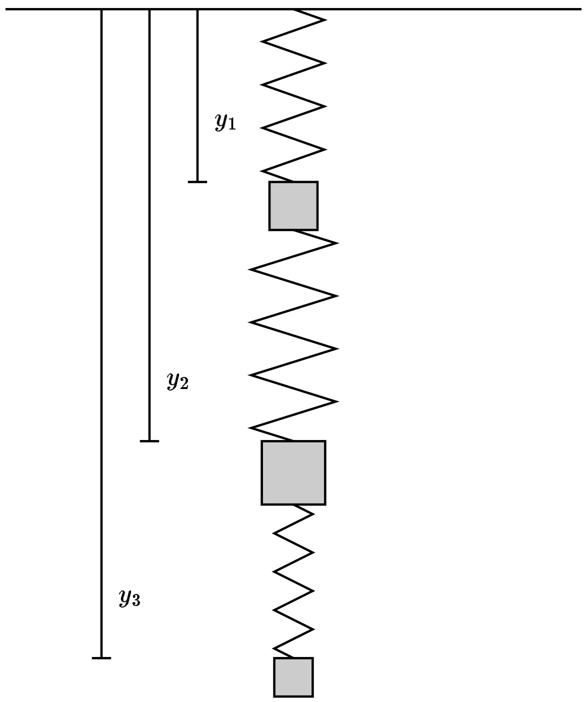
We are going to generalize the system we looked at earlier, where one weight was hung from a moving support. In the new situation, we shall hang n weights in a series from n springs, the whole system being suspended from a moving support.

What are the components of this system? (1) We have the weights with mass m_i, which we take to be small blocks of height h_i. (2) We have the springs of length ℓ_i and spring constant k_i. (3) We have the point of attachment at the top. Fix a certain base level which we shall say to be at height 0. All other motion will be measured from this fixed level, measured positively in the downward direction. We want the point of attachment possibly to move up and down—let $d_0(t)$ be its level at time t. It is the motion of this top support which will drive the motion of the weights.

Let d_i be the displacement of the top of the i-th weight from the fixed level. Let s_i be the stretch in the i-th spring from its relaxed position. According to Hooke's Law, the force exerted by the spring is then of magnitude $k_i s_i$ exterted inwards along the spring. There is a simple relationship between the s_i and the d_i. The top of the i-th spring is at displacement d_i. As we go down from the top of the i-th weight, we traverse first the height h_i of the i-th weight. Then we cover the total extended length of the $i+1$-st spring, which is $\ell_{i+1} + s_{i+1}$, to get to the top of the next weight. Therefore

$$d_{i+1} = s_{i+1} + \ell_{i+1} + h_i + d_i \ .$$

This works even for $i = 0$ if we assume an imaginary 0-th weight with no size at all, or $h_0 = 0$.



On the point at the top of the i-th weight there are three forces: (1) a pull up equal to $k_i s_i$; (2) the weight $m_i g$ pulling down; (3) the pull down of the $i+1$-st spring, equal to $k_{i+1} s_{i+1}$ (equal to 0 at the bottom weight). The only tricky thing here is that the total effect of the weights below the i-th cannot be transmitted except through the $i+1$-st spring. This is an example of a very general principle which asserts that in all of nature *forces are never transmitted instantly at a distance, but must have a mode of transmission*. In this example, it is the springs which transmit the forces.

The total force on the i-weight is

$$-k_i s_i + m_i g + k_{i+1} s_{i+1} \ .$$

We now want an expression for the force in terms of our coordinates d_i, or something equivalent to it. We could substitute for the s_i in terms of the d_i directly, but the expressions we get are somewhat messy. We can simplify things by introducing some new variables. Let $d_{i,\mathrm{relaxed}}$ be the heights d_i in the configuration when the springs are not stretched at all. In this configuration $s_i = 0$. Thus

$$d_{i+1,\mathrm{relaxed}} = d_{i,\mathrm{relaxed}} + h_i + \ell_{i+1} \ .$$

If we compare this directly to the previous equation

$$d_{i+1} = s_{i+1} + \ell_{i+1} + h_i + d_i \ .$$

and set $x_i = d_i - d_{i,\mathrm{relaxed}}$ for $i \geq 1$ and $x_0 = d_0$ (the displacement of the support), we get

$$d_{i+1} - d_{i+1,\mathrm{relaxed}} = (d_i - d_{i,\mathrm{relaxed}}) + s_{i+1}, \qquad s_{i+1} = x_{i+1} - x_i \ .$$

to get the forces on the i-th weight equal to

$$\begin{aligned} F_i &= -k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) + m_i g \\ &= k_i x_{i-1} - (k_i + k_{i+1}) x_i + k_{i+1} x_{i+1} + m_i g \ . \end{aligned}$$

In the equilibrium position, the forces all vanish, and the support has zero displacement from the base level. We get equations

$$\begin{aligned} -k_1x_1 + k_2(x_2 - x_1) + m_1 g &= 0 \\ -k_2(x_2 - x_1) + k_3(x_3 - x_2) + m_2 g &= 0 \\ &\dots \\ -k_n(x_n - x_{n-1}) + m_n g &= 0 \end{aligned}$$

or

$$\begin{aligned} (k_1 + k_2)x_1 - k_2 x_2 &= m_1 g \\ -k_2 x_1 + (k_2 + k_3)x_2 - k_3 x_3 &= m_2 g \\ &\dots \\ -k_n x_{n-1} + k_n x_n &= m_n g \end{aligned}$$

which turn out to have a relatively simple solution for the equilibrium position.

For example, if $n = 2$ we get equations

$$\begin{aligned} (k_1 + k_2)x_1 - k_2 x_2 &= m_1 g + k_1 x_0 \\ -k_2 x_1 + k_2 x_2 &= m_2 g \end{aligned}$$

If we add these two equations we get

$$k_1 x_1 = m_1 g + m_2 g$$

from which we can solve for x_1, then solve in turn for x_2.

From now on, let $x_{i,\mathrm{eq}}$ be the values of x_i in the equilibrium position.

Exercise 1.1. *Solve for the equilibrium values of x_i if $n = 3$. (Hint: add the equilibrium equations up from the bottom. You get a particularly simple set of equations. Explain this from basic principles.)*

Exercise 1.2. *Simplify the equations for general n.*

Let K be the **stiffness matrix**

$$\begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \dots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & \dots & -k_n & k_{n-1} \end{bmatrix}$$

The equilibrium equations become

$$K \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} m_1 g \\ m_2 g \\ \dots \\ m_n g \end{bmatrix}$$

Exercise 1.3. *Write down the equlibrium matrix and the equations for equilibrium for 3 weights on 3 springs.*

Exercise 1.4. *Find formulas for the equilibrium values of x_i when $n = 3$.*

Exercise 1.5. *Do an analysis of the situation when the weights are tied down at the bottom by an $n + 1$-st spring. In particular, write down the analogue of K.*

2. A pair of weights on a spring

Suppose now we consider just a pair of weights, one hanging below the other. This is not only the next simplest step up from the case of a single weight, but almost all of the phenomena which we meet in large systems already occur here.

We shall in this case review what we said in the previous section. We must first analyze forces on the weights, and then use Newton's second law to relate these forces to motion.

We shall assume that the springs are not stretched out of shape, so that Hooke's law holds. Recall that the forces exerted by a spring are then proportional to the elongation of the spring, and that the constant of proportionality is called the **spring constant**.

Let m_1 and m_2 be the masses of the springs, k_0 and k_1 the spring constants for each of the springs involved.

Suppose the weights are in any configuration where x_0 and x_1 are the displacements of the tops of the weights from the positions where the springs are relaxed. Then the elongation in the first spring is x_0, that in the second $x_1 - x_0$. (If both weights are displaced the same distance, there is no stretch in the spring between them.) The force acting at the top of the first weight is (1) a pull up by the first spring equal to $-k_0 x_0$; (2) the first weight of magnitude $m_1 g$ pulling down; (3) the tension in the second spring $k(x_1 - x_0)$. (The second weight is transmitted through the lower spring and does not have to be counted separately.) The total force at the top of the first weight is therefore

$$F_1 = -k_0 x_0 + m_1 g + k_1(x_1 - x_0) \ .$$

What is the force F_2 at the top of the second weight? (1) $-k_1(x_1 - x_0)$ exerted by the lower spring upwards; (2) the weight $m_2 g$ downwards. Total:

$$F_2 = -k_1(x_1 - x_0) + m_2 g \ .$$

At equilibrium these forces must vanish. Let $x_{i,\mathrm{eq}}$ be the elongations at equilibrium. We get a pair of equations

$$\begin{aligned} 0 &= -k_0 x_{\mathrm{eq},0} + m_1 g + k_1(x_{\mathrm{eq},1} - x_{\mathrm{eq},0}) \\ 0 &= -k_1(x_{\mathrm{eq},1} - x_{\mathrm{eq},0}) + m_2 g \end{aligned}$$

which can be solved, if necessary, for the $x_{i,\mathrm{eq}}$.

If the system is not in equilibrium there will be motion, according to Newton's law. We get equations

$$\begin{aligned} m_1 x_0'' &= -k_0 x_0 + m_1 g + k_1(x_1 - x_0) \\ m_2 x_1'' &= -k_1(x_1 - x_0) + m_2 g \ . \end{aligned}$$

Write $x_i = x_{\mathrm{eq},i} + y_i$ so that y_i is the displacement of the weights from equilibrium. If we substitute these expressions into the equations we get

$$\begin{aligned} m_1\, y_0'' &= -k_0[x_{\mathrm{eq},0} + y_0] + m_1 g + k_1([x_{\mathrm{eq},1} + y_1] - [x_{\mathrm{eq},0} + y_0]) \\ m_2\, y_1'' &= -k_1([x_{\mathrm{eq},1} + y_1] - [x_{\mathrm{eq},0} + y_0]) + m_2 g \ . \end{aligned}$$

The terms involving the weights and equilibrium positions all cancel out, and we can rewrite these as

$$\begin{aligned} m_1\, y_0'' &= -k_0 y_0 + k_1(y_1 - y_0) \\ m_2\, y_1'' &= -k_1(y_1 - y_0) \end{aligned}$$

or

$$\begin{aligned} m_1\, y_0'' &= -(k_0 + k_1) y_0 + k_1 y_1 \\ m_2\, y_1'' &= k_1 y_0 - k_1 y_1 \ . \end{aligned}$$

This is a **system of second order differential equations** in the two unknown functions y_0 and y_1. The fact that the right hand side of each equation involves both y_0 and y_1 means that the two functions y_0 and y_1 are **coupled** to each other. The two weights do not move independently of each other, and it is in fact a bit difficult to imagine how they do move. At any rate, in this case the equations can be put in **matrix form**

$$\begin{bmatrix} m_1\, y_0'' \\ m_2\, y_1'' \end{bmatrix} = \begin{bmatrix} -(k_0 + k_1) & k_1 \\ k_1 & -k_1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \ .$$

We expect the motion of the pair of weights to be a vibration. With what frequencies? If we look for periodic motion we set

$$y_0 = C_0 \cos \omega t, \quad y_1 = C_1 \cos \omega t$$

and get

$$\begin{aligned} -m_1 C_0 \omega^2 \cos \omega t &= -k_0 C_0 \cos \omega t + k_0 (C_1 \cos \omega t - C_0 \cos \omega t) \\ -m_2 C_1 \omega^2 \cos \omega t &= -k_1 (C_1 \cos \omega t - C_0 \cos \omega t) \ . \end{aligned}$$

This has to hold for all values of t. We can therefore cancel the term $\cos \omega t$ and get

$$\begin{aligned} -m_1 C_0 \omega^2 &= -k_0 C_1 + k_1 (C_1 - C_0) \\ -m_2 C_1 \omega^2 &= -k_1 (C_1 - C_0) \end{aligned}$$

or

$$\begin{aligned} (k_0 + k_1 - m_1 \omega^2) C_0 - k_1 C_1 &= 0 \\ -k_1 C_0 + (k_1 - m_2 \omega^2) C_1 &= 0 \ . \end{aligned}$$

which is a pair of homogeneous linear equations in two unknowns C_0 and C_1.

We can write it in matrix terms

$$\begin{bmatrix} k_0 + k_1 - m_1 \omega^2 & -k_1 \\ -k_1 & k_1 - m_2 \omega^2 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

One solution is just $C_0 = C_1 = 0$, but this means there is no motion. If we have any motion at all there will exist a some non-trivial solution of the pair. Here is the key point, which I recall from linear algebra:

- *A pair of homogeneous equations in two unknowns can have a non-trivial solution if and only if the coefficient matrix is singular, which means that its determinant vanishes.*

This imposes a condition on ω. Incidentally, the reason this principle holds is very simple: if the determinant were not zero then the coefficient matrix would have an inverse, and we could solve the system uniquely.

For the moment, for simplicity, I shall assume that the two masses are the same:

$$m_1 = m_2 = m \ .$$

The coefficient matrix is then

$$\begin{bmatrix} k_0 + k_1 - m \omega^2 & -k_1 \\ -k_1 & k_1 - m \omega^2 \end{bmatrix} = K - m \omega^2 \, I$$

if

$$K = \begin{bmatrix} k_0 + k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \ .$$

The determinant of the coefficient matrix vanishes if and only if

$$\det(K - m \omega^2 \, I) = 0$$

which means that $m\omega^2$ is an **eigenvalue** of the matrix K. The components (C_0, C_1) turn out to be the coefficients of the corresponding eigenvectors, and describe the **modes of motion** of the system when it is vibrating with a single frequency.

This is the way it always happens.

- *Frequencies of motion of complicated systems and the modes of motion at those frequencies are related directly to eigenvalues and eigenvectors.*

More generally:

- *Solving a system of linear differential equations is roughly equivalent to finding eigenvalues and eigenvectors of the coefficient matrix.*

We shall review eigenvalues and eigenvectors of matrices later on.

Example. Suppose $m_1 = m_2 = 1$, $k_0 = 3$, $k_1 = 2$. Set $\omega^2 = \lambda$. The coefficient matrix is

$$\begin{bmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{bmatrix} .$$

The determinant is

$$(5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 6$$

so we get the equation

$$\lambda^2 - 7\lambda + 6 = 0$$

with solutions

$$\lambda = 1, 6$$

so that the possible frequencies are

$$\omega = 1, \sqrt{6} .$$

In other words, there are exactly two possible frequencies of motion for the system.

The eigenvectors for 1 and 6 are then

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -1 \end{bmatrix} .$$

We get the motions

$$\begin{bmatrix} \cos t \\ 2\cos t \end{bmatrix} \text{ and } \begin{bmatrix} 2\cos\sqrt{6}t \\ -\cos\sqrt{6}t \end{bmatrix} .$$

As you can verify, the motions

$$\begin{bmatrix} \sin t \\ 2\sin t \end{bmatrix} \text{ and } \begin{bmatrix} 2\sin\sqrt{6}t \\ -\sin\sqrt{6}t \end{bmatrix} .$$

are also possible solutions of the system. The system is linear and homogeneous, and any combination of these will also be possible motions. In fact any possible motion will be such a linear combination

$$c_1 \begin{bmatrix} \cos t \\ 2\cos t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ 2\sin t \end{bmatrix} + c_3 \begin{bmatrix} 2\cos\sqrt{6}t \\ -\cos\sqrt{6}t \end{bmatrix} + c_4 \begin{bmatrix} 2\sin\sqrt{6}t \\ -\sin\sqrt{6}t \end{bmatrix} .$$

Any linear combination of the first two has frequency 1 and any linear combination of the second two has frequency $\sqrt{6}$. An arbitrary solution will therefore be a sum of two motions with very different frequencies, and its motion will be in general be very complicated. It happens here, for reasons that I will explain a bit later on, that the eigenvectors here are perpendicular to each other. An interesting consequence of this perpendicularity is that the energy of any possible motion is the sum of the energies of its two periodic components.

Remark. The pair of equations

$$\begin{bmatrix} m_1\, y_0'' \\ m_2\, y_1'' \end{bmatrix} = \begin{bmatrix} -(k_0+k_1) & k_1 \\ k_1 & -k_1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

can also be written in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} y_0'' \\ y_1'' \end{bmatrix} = \begin{bmatrix} -(k_0+k_1) & k_1 \\ k_1 & -k_1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

or

$$My'' = -Ky$$

where M is the **mass matrix**, K the **stiffness matrix**:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad K = \begin{bmatrix} (k_0+k_1) & -k_1 \\ -k_1 & k_1 \ . \end{bmatrix}$$

This can be thought of as a matrix version of Hooke's law, with matrices M and K replacing the scalars m and k. In other words, the matrix equation $F = -Ky$ still asserts that stress (force) is proportional to strain (displacement from equilibrium) but stress and strain here have several different components. The equation $F = My''$ is a matrix version of Newton's second law, where F is a force with several components.

If we try to find a periodic motion we set $y = (\cos \omega t)\, y_0$ where y_0 is a constant vector, not identically $(0,0)$. We arrive at the matrix equations

$$(K - \omega^2 M)y_0 = 0, \quad \det(K - \omega^2 I) = 0 \ .$$

If the masses are the same we arrive at the eigenvector / eigenvalue equation as before. Without that assumption we are looking at **generalized** eigenvectors and values. Finding them reduces easily to finding ordinary eigenvectors and eigenvalues.

- *In realistic physical systems, finding vibration frequencies amounts to solving a generalized eigenvalue problem.*

The eigenvectors determine the **modes** of vibration, that is to say the exact way in which different components will move in purely periodic motion. In practice, this can be very important—if one component of motion is very large at some particular frequency, it may be overstressed. One aim of good structure design is to control eigenvalues and eigenvectors so that vibrations of probable frequencies will not stimulate dangerous modes of motion.

Exercise 2.1. *Write down the equations for the vibration frequencies in both cases (n springs, $n+1$ springs). Use the* **mass matrix** *which if $n = 3$ is equal to*

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} .$$

Exercise 2.2. *Write down the matrices K and M if there are three weights hanging on three springs. If there are n weights on n springs?*

Exercise 2.3. *Suppose we have two weights both of mass 1 and that $k_0 = 2$, $k_1 = 4$. What are equilibrium displacements, if our units are such that $g = 1$?*

Exercise 2.4. *Suppose we have two weights both of mass 1 and that $k_0 = 2$, $k_1 = 4$. What are the frequencies of vibration?*

Exercise 2.5. *Suppose we have two weights of masses 1 and 2, that $k_0 = 1$, $k_1 = 3$. What are the frequencies of vibration?*

2. Another way to find the differential equations for large systems

We used Newton's second law applied to each of the weights to obtain the pair of second order differential equations determining the evolution of the system. It was not too easy to see exactly how forces worked out, and for more complicated systems this sort of analysis becomes almost impossible. We can find a formulation of Newton's law, however, which uses directly an expression for energy in the system.

For a single weight on a spring there are two sources of potential energy in the system and one of kinetic energy. (1) The potential energy of the weight itself depends on its height. There is some ambiguity in where we measure the height from, but potential energy is only defined up to an integration constant anyway. This component of potential energy we can measure as $-mgx$ where x is displacement (down) from the relaxed position of the spring. (2) The potential energy in the spring under tension, which is $kx^2/2$. (3) The kinetic energy which is

$$\mathrm{KE} = mv^2/2$$

where $v = x'$. We know that the potential energy which is

$$\mathrm{PE} = -mgx + kx^2/2$$

takes a minimum value at equilibrium. If we take the derivative with respect to x and set it equal to 0 we get

$$-mg + kx_{\mathrm{eq}} = 0 \ .$$

If we are not in equilibrium then x will change in time. Newton's law $F = ma$ becomes

$$\frac{d}{dt}\,\frac{\partial\,\mathrm{KE}}{\partial v} = -\frac{\partial\,\mathrm{PE}}{\partial x} \ .$$

Since

$$\frac{\partial\,\mathrm{KE}}{\partial v} = mv, \quad \frac{\partial\,\mathrm{PE}}{\partial x} = -mg + kx$$

this translates to the same equation we obtained before:

$$\begin{aligned} mv' &= mg - kx \\ &= mg - kx_{\mathrm{eq}} - k(x - x_{\mathrm{eq}}) \\ my'' &= -ky \end{aligned}$$

where $y = x - x_{\mathrm{eq}}$.

This generalizes in the following way: If we have a system with n degrees of freedom, say where the state is determined by position variables x_i and velocity variables v_i then the system of differential equations determining the evolution of the system is the set

$$\frac{d}{dt}\,\frac{\partial\,\mathrm{KE}}{\partial v_i} = -\frac{\partial\,\mathrm{PE}}{\partial x_i} \quad (i = 1 \ldots n)$$

where after calculating partial derivatives we substitute $v_i = x_i'$. Using this is often easier than trying to calculate forces and use Newton's law directly. In the special case of an equilibrium solution all velocities are 0 and we recover the fact that in equilibrium the derivative of potential energy with respect to all position coordinates vanishes.

In the case of two weights, recall that the coordinates are the displacements of the weights from the positions where the springs are relaxed. The elongation of the first spring is x_0, the second is $x_1 - x_0$. The kinetic energy is

$$\mathrm{KE} = m_1 v_1^2/2 + m_2 v_2^2/2$$

and the potential energy is that of the weights and springs separately:

$$\mathrm{PE} = (1/2)k_0 x_0^2 + (1/2)k_1(x_1 - x_0)^2 - m_1 g x_1 - m_2 g x_2 \ .$$

We get equations

$$\begin{aligned} m_1 v_0' &= m_1 g - m_1 x_0 + m_2(x_1 - x_0) \\ m_2 v_1' &= m_2 g - m_2(x_1 - x_0) \end{aligned}$$

which agree with what we got before if we let $y_i = x_i - x_{i,\mathrm{eq}}$ and recall that at equilibrium the partial derivatives of potential energy are 0.

The advantage of this technique is that it is often relatively simple to calculate the energy as a sum of individual components, even in large systems, where to keep track of how the forces act can be apparently more difficult.

Exercise 2.1. *Write down a formula for kinetic and potential energy if there are three weights.*