Chapter 1. How things cool off

One physical system in which many important phenomena occur is that where an initial uneven temperature distribution causes heat to flow. Of course as heat flows the temperature distribution changes, which in turn modifies the heat flow. That is to say: Heat flows from hot places to cold ones, and as this happens, the temperature of cold places rises and the temperature of hot places decreases. There are two rough mathematical rules governing the relationship between heat flow and temperature change:

- *Heat flow is proportional to spatial rates of change in temperature distribution.*
- *The time rate of temperature change at any point is proportional to the rate of heat flow into that point.*

1. Newton's law of cooling

The general situation can be really complicated, and we shall look at it in some detail later on. But right now we shall look at a simple situation, that where a small object is cooling off or heating up in a large environment of fixed temperature. The point of assuming that the object is small is that the temperature inside it will be pretty much uniform, and in fact we shall assume the temperature to be constant throughout it.

The object is cooling off because heat is flowing into or out of it from the environment, and the heat flow takes place because of a temperature difference between the object and the environment.

The rate at which a warm object in a cold room cools off depends on how hot it is. Not only does a hot object cool off, but the hotter it is the more rapidly it cools. A somewhat more precise attempt to describe this behaviour is

- **Newton's law of cooling.** *The rate at which an isolated object changes temperature is proportional to the difference between its temperature and the surrounding temperature.*

If $\theta(t)$ is the temperature of the object at time t and $\theta_{\rm env}$ is the temperature of its environment, Newton's law can be written

$$\theta' = -k(\theta - \theta_{\rm env})$$

with k a positive constant. In other words, illustrating one of the basic principles of this course:

- *The natural mathematical expression of Newton's law of cooling is a differential equation of first order.*

One important thing to realize is that the constant of proportionality depends only on the object and the environment. In general the environmental temperature $\theta_{\rm env}$ will depend on time. We might be looking at how things cool outside, for example, where temperature oscillates back and forth from day to night. But for the moment we shall just look at the simplest case when it is held constant. Intuitively, in this case we expect the object's temperature simply to approach $\theta_{\rm env}$ as time goes on. Suppose, for example, that $\theta_{\rm env} = 0$. Then Newton's Law becomes

$$\frac{d\theta}{dt} = \theta' = -k\theta \ .$$

This is an equation you have seen in your calculus course. We can solve it by dividing both sides by θ and then rearranging to get

$$\frac{d\theta}{\theta} = -k\,dt \ .$$

If we integrate both sides we get

$$\log\theta = -kt + c$$

for some constant of integration c or

$$\theta = Ce^{-kt}$$

for some other constant $C = e^c$. **Here and throughout these notes,** $\log x$ **will be the same as** $\log_e x$ **or** $\ln x$. If we set $t = 0$ we see that C must be the initial temperature $y(0)$ of the object. In other words, in this environment the temperature will decrease exponentially towards 0.

Whether $\theta_{\text{env}} = 0$ or not cannot be significant, since the level at which zero temperature is set is rather arbitrary. To solve the general equation we rewrite it as

$$(\theta - \theta_{\text{env}})' = -k(\theta - \theta_{\text{env}})$$

which amounts to a shift in origin to the environmental temperature. Then we get the solution

$$\theta - \theta_{\text{env}} = Ce^{-kt}, \quad \theta = \theta_{\text{env}} + Ce^{-kt}$$

for some constant C. The constant C is determined by the conditions in which things started out. Suppose the initial temperature at $t = 0$ to be θ_0. If we set $t = 0$ then the formula gives us $\theta_0 = \theta_{\text{env}} + C$, so that

$$C = \theta_0 - \theta_{\text{env}},$$

the initial temperature difference.

- *The solution to the differential equation*

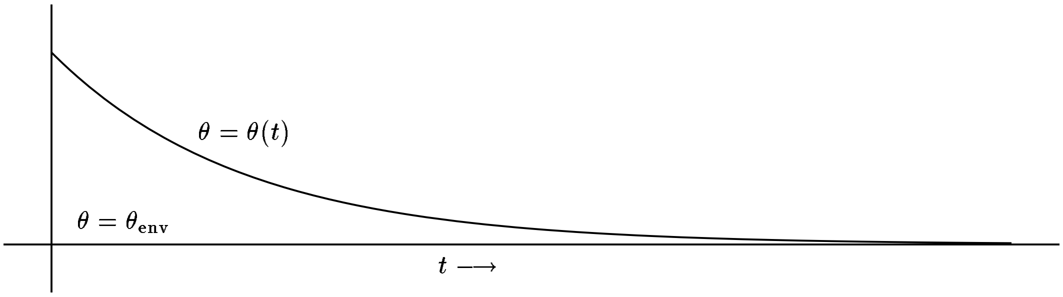
  $$\theta' = -k(\theta - \theta_{\text{env}})$$

  where θ_{env} is a constant, and where $\theta(0) = \theta_0$, is

  $$\theta = \theta_{\text{env}} + (\theta_0 - \theta_{\text{env}})e^{-kt} \ .$$

This says that *the difference between the temperature of the object and the surrounding object decreases at an exponential rate.*

If the initial temperature is the same as the environmental temperature, it doesn't change. In other words, the constant temperature function $\theta(t) = \theta_{\text{env}}$ is itself a solution of the differential equation. It is called an **equilibrium** solution of the equation.



This sort of approach to equilibrium in physical systems is very common. It is characterized here by the property that *in any fixed amount of time the difference between temperature and equilibrium is multiplied by a constant factor.* To be precise, in a time interval Δt the temperature difference scales by $e^{-k\,\Delta t}$. The larger k is, the more rapidly the object cools off. To be precise, the time it takes to cool off is inversely proportional to k.

- *The amount of time it takes for the difference to scale by the factor e^{-1} is called the* **relaxation time** *τ of the system.*

If we solve $e^{-k\,\Delta t} = e^{-1}$ we see that

$$\Delta t = \tau = \frac{1}{k}$$

and we can rewrite the formula for θ as

$$\theta(t) = (\theta_0 - \theta_{\text{env}})e^{-t/\tau} + \theta_{\text{env}} \ .$$

Exercise 1.1. *A cup of coffee initially at 90° sitting in a room at 20° takes 5 minutes to cool to 70°. (a) What is the coffee temperature after 10 minutes? (b) What is the relaxation time? (c) How long does it take to cool to 40°? (d) Write down a formula for the temperature at time t (given in minutes).*

2. The mathematics abstracted

What we have said is not tied necessarily to the physics of a cooling object. There are many circumstances in which we want to solve a first order differential equation

$$y' = ay + b$$

where a and b are constants. We can rewrite this equation as

$$\frac{dy}{ay+b} = dt$$

and integrate both sides, substituting $u = ay + b$, to get

$$\begin{aligned} \frac{du}{au} &= dt \\ \log(ay+b) &= at + c \\ ay + b &= e^c e^{at} \\ y &= Ce^{at} - \frac{b}{a} \end{aligned}$$

for some suitable constant C. If we set $t = 0$, we get

$$y_0 = C - \frac{b}{a}$$

so $C = y_0 + b/a$.

- *The solution to*

  $$y' = ay + b, \quad y(0) = y_0$$

  is

  $$y = (y_0 + b/a)e^{at} - b/a = y_0 e^{at} + \left(\frac{b}{a}\right)(e^{at} - 1)$$

We shall see in a later section that if we look at an object cooling in an environment whose temperature is changing with time, we arrive at a differential equation of the form

$$y' = ay + b(t)$$

where now $b(t)$ might depend on time. We can find the general solution of this equation by a trick. Rewrite it as

$$y' - ay = b(t) \ .$$

Multiply both sides by e^{-at} to get

$$y'e^{-at} - aye^{-at} = e^{-at}b(t) \ .$$

Notice that the left hand side is the derivative of ye^{-at} according to the product rule for derivatives. Therefore we have

$$\frac{d\,ye^{-at}}{dt} = e^{-at}b(t)$$

which means that $e^{-at}y$ is the indefinite integral of $e^{-at}b(t)$. In a slightly pedantic formulation

$$ye^{-at} = \int^t e^{-as}b(s)\,dt, \quad y = e^{at}\int^t e^{-as}b(s)\,ds \ .$$

We need the variable s, which is called a *dummy* variable or *variable of integration* in calculus, because it *must* be distinguished from t. The integral here is indefinite. If we take an integration constant into account we can write this formula a bit more suggestively:

- *The general solution to the differential equation*

$$y' = ay + b(t)$$

is

$$y = Ce^{at} + e^{at} \int^t e^{-as} b(s)\, ds \ .$$

where C is an arbitrary constant.

The integral here is still indefinite, undetermined up to an additive constant. Any such constant will affect the constant C, so that we cannot expect C to have any particular meaning unless we fix that integration constant. We can do this by changing the indefinite integral into a definite one:

$$y = Ce^{at} + e^{at} \int_{t_0}^t e^{-as} b(s)\, ds$$

where t_0 is some arbitrarily chosen moment of time. Now if we set $t = t_0$ the integral becomes 0 and we get $y(t_0) = Ce^{at_0}$ which gives

$$C = e^{-at_0} y(t_0)$$

and we have:

- *The solution to the equation*

$$y' = ay + b(t)$$

such that $y(t_0) = y_0$ is given by the formula

$$y = y_0 e^{a(t-t_0)} + e^{at} \int_{t_0}^t e^{-as} b(s)\, ds \ .$$

Exercise 2.1. *A radioactive substance emits radiation and changes to a new substance. The amount of radioactivity is proportional to the quantity of the original substance remaining. The half-life of a radioactive substance is the amount of time it takes for half the substance to decay radioactively. If the half life of radium is 1760 years, how much radium is left from an initial gram of radium after 100 years?*

Exercise 2.2. *What is the exact relationship between half-life and relaxation time?*

Exercise 2.3. *Solve the differential equation & initial conditions*

$$y' + y = t, \quad y(0) = 0 \ .$$

3. When the right hand side is an exponential function

In many practical cases, $b(t) = e^{ct}$. In this case the solution of $y' = ay + e^{ct}$ has a particularly simple form. We apply the integral formula to get

$$\begin{aligned} y &= Ce^{at} + e^{at} \int^t e^{-as} e^{cs}\, ds \\ &= Ce^{at} + e^{at} \int^t e^{(c-a)s}\, ds \\ &= Ce^{at} + e^{at} \frac{e^{(c-a)t}}{c-a} \quad (c \neq a) \\ &= Ce^{at} + \frac{e^{ct}}{c-a} \\ &= Ce^{at} + te^{at} \quad (c = a) \end{aligned}$$

Exercise 3.1. *Suppose a cup of coffee is cooling off in a room which is itself cooling off. Say that the relaxation time for the coffee in the room is 20 minutes, and that the room cooling has a relaxation time of 6 hours. Suppose the coffee is initially at 100°, and the room initially at 25° cooling off to an outside temperature of 0°. What are the temperatures of the coffee and the room after one hour?*

Exercise 3.2. *Suppose we have two radioactive substances A and B. The substance A decays into B with a relaxation time of τ_A, and B decays into the stable substance C with relaxation time τ_B. Let y_A, y_B, y_C be the amounts of each substance (depending on time). Write down three first order differential equations describing the relationship between the three quantities.*

Exercise 3.3. (Continuing) *Check your answer by verifying that the total amount of all three doesn't change.*

Exercise 3.4. (Continuing) *Assuming that we start with an amount y_0 of substance A at $t = 0$ and no other substances, find an expression for y_A and y_B at time t.*

4. Separable first order equations

We have seen above one technique for solving first order differential equations which works rarely, but which occurs often enough to call attention to it. We solved the equation

$$\frac{dy}{dt} = y' = ay + b$$

by rewriting it as

$$\frac{dy}{ay + b} = dt$$

and then integrating both sides. When we can rewrite a first order equation

$$y' = f(t, y)$$

so all occurrences of y are on one side and all occurrences of t are on the other, the equation is said to be **separable**. We can then integrate to find an implicit equation for y in terms of t. Usually this equation does not tell us much that is practical, even when it can be found.

Exercise 4.1. *An object at radius r falling towards Earth satisfies the first order differential equation*

$$r' = -\sqrt{\frac{2gR^2}{r} + \frac{2E}{m}}$$

where E is the (constant) energy of the object. Solve this by separation of r and t, integrate what you get (by substitution), and find an implicit equation for r in terms of t.

Exercise 4.2. *The rate of evaporation of a spherical raindrop is proportional to its surface area. If it takes 5 minutes to evaporate from 3 mm. radius to 2 mm. radius, how much longer does it take to disappear?*

Exercise 4.3. *Find a formula for general solution of*

$$y' = a(t)y\ .$$

Exercise 4.4. *By mimicking the derivation for the solution of $y' = ay + b(t)$, find a formula for the general solution of*

$$y' = a(t)y + b(t)\ .$$