Introduction

If we drop an object from height h_0 near the Earth's surface, then (neglecting air friction) it accelerates at a constant acceleration $g = 980\ \mathrm{cm/sec}^2$. At time t after release its velocity is gt, and the distance it has fallen is

$$\int_0^t v\, ds = \int_0^t gs\, ds = gt^2/2$$

so its height will be $h(t) = h_0 - gt^2/2$. This rule was first discovered by Galileo. If we want to know how long it takes to drop to height h, we solve the equation for $h(t)$ to get

$$t = \sqrt{\frac{2\,(h_0 - h)}{g}}\ .$$

Things are more complicated if we drop the object from a height far above the Earth's surface. In this case, we must take into account the fact that the force of gravity weakens as the distance from Earth increases. This means that as the object falls, its acceleration will change. To be exact, gravitational acceleration is inversely proportional to the square of the distance r from the centre of the Earth, and acceleration at radius r is

$$g_r = -\frac{k}{r^2} = -\frac{g}{(r/R)^2}$$

(so that $k = gR^2$) where g is the gravitational constant at the surface of the Earth and R is the Earth's radius. For example, at $r = 10,000$ kilometers we have a gravitational acceleration of

$$g_{10,000} = (9.80)(6370/10000)^2 = 3.98\ \mathrm{m/sec}^2$$

since R is about $6370\ \mathrm{km}$. To deal with the new scale of things, we change units from meters to kilometers. We then combine the formula for gravitational force with Newton's Second Law $F = ma$ where a is the acceleration $a = r''$, to obtain an equation

$$r'' = -\frac{0.00980}{(r/R)^2}$$

which we can rewrite as

$$r'' = -\frac{k}{r^2} = -\frac{397654}{r^2}\ .$$

This is a **differential equation** describing the way in which r changes as a function of time. It is said to be of **second order** because it involves the second derivative r''. The argument we have made here is quite common, and offers a particular case of one of the basic principles relating mathematics to the real world:

- *The laws of physics, when translated directly into mathematics, are frequently expressed as differential equations.*

What good does it do to know this differential equation? For example, what can we now say about what happens if we drop an object from a distance of $10,000$ km from the Earth's centre? *How long, for example, does it take to hit the Earth's surface?* It is possible to get an answer on theoretical grounds and some clever manipulations, but more important to us is an intuitive approach to this question.

We can get some idea of how things go by making some easy if tedious calculations. Lots of them. The basic idea of the calculation is this:

> *Suppose that at time t we are at radius r with velocity v. These variables t, r, v describe completely the* **state** *of the falling object, which is to say that as far as we are concerned here they determine the subsequent behaviour of the object. The acceleration a at time t is then $-k/r^2$. Suppose a small amount*

of time Δt passes. The definitions of velocity and acceleration tell us how to find an approximation for the state at time $t + \Delta t$—in time Δt the object falls approximately a distance $v\,\Delta t$, and its velocity increases by $a\,\Delta t$. The new radius will therefore be approximately $r + v\,\Delta t$ and the new velocity will be approximately $v + a\,\Delta t$. We can now repeat the same set of calculations over again for the next time interval. This suggests that we start with an initial state $r_0 = 10000$ and $v_0 = 0$, choose some reasonably small time interval Δt, and calculate a sequence of approximate states (r_n, v_n) at time $n\,\Delta t$ by the rules

$$\begin{aligned} a_n &= -397654/r_n^2 \\ r_{n+1} &= r_n + v_n\,\Delta t \\ v_{n+1} &= v_n + a_n\,\Delta t\ . \end{aligned}$$

Of course the approximation will be better if we use smaller time intervals. In this case, carrying out the calculations by hand would be a ridiculous amount of work—so we do it by computer. This illustrates a second fact about differential equations, and another of the basic principles of this course:

- *Frequently, the best way to solve a differential equation is by computer calculation.*

In our case, here is a selection from the output of the computation with $\Delta t = 1$ second:

t	h	v	a
0	10000.000	0.0000	−0.00398
1	10000.000	−0.0040	−0.00398
2	9999.996	−0.0080	−0.00398
...			
1264	6377.130	−6.7226	−0.00978
1265	6370.408	−6.7324	−0.00980
1266	6363.676	−6.7422	−0.00982

so that it seems to take a little more than 1265 seconds to fall from a height of $10,000$ km to Earth.

It is not my point here, however, to give you a practical method for calculating approximations to the solutions of a differential equation. Instead, I hope it will give you a feel for the **essential nature of a differential equation**. The simple calculations suggested above encapsulate this nicely:

- *A differential equation is a description of how the state of a physical system changes instantaneously.*

However, this is not the whole story. Relatively little of this course will be concerned with numerical calculation. Instead, we shall look at a number of interesting examples of differential equations which can in fact be solved exactly. These will often serve as models in understanding what can happen more generally.

- *In order to understand what the computer tells us, we shall need to have a stock of simple models to call on, which we do in fact know how to solve exactly.*

In the case of a falling object, for example, if we want to see if the computer is producing reasonable results we might compare the calculations to Galileo's simpler case, at least for small values of t.

We shall come back later to the question of how to use a computer to approximate the solutions of differential equations, and we shall return later also to look at differential equations of second order. But first we shall look at some differential equations of first order modeling simple physical processes.

Exercise 1. *At time $t = 200$, the object dropped from $10,000$ km has approximately*

$$\begin{aligned} r &= \quad 9920.66 \text{ km} \\ v &= -0.7995 \text{ km/sec} \end{aligned}$$

Tell approximately what its height and velocity are at time $t = 201$.

Exercise 2. *At $t = 1265$, $r = 6370.408$, while at $t = 1266$ it is at $r = 6363.676$. Calculate the moment of contact to within 0.1 second.*

Exercise 3. *Suppose that gravitational acceleration at $10,000$ km were the same as it is at the Earth's radius. How long would it take our object to fall to the surface?*

Exercise 4. *Suppose that the acceleration during all of the fall were what it is at $10,000$ km. How long would the fall take?*

Exercise 5. *Suppose that near the Earth's surface an object is thrown down from height h_0 with speed v_0. What height is it at time t? How long to fall a distance Δh?*

Exercise 6. *The differential equation in this section is rather special, since energy is conserved as the object falls. The kinetic energy is $mv^2/2$ and the potential energy is $-mk/r$ (a negative coefficient because the object loses potential energy as it falls). Recall that potential energy is defined up to some fixed constant, and here that constant is chosen to make the potential energy vanish at $r = \infty$. (a) Write down the expression for total energy. (b) Differentiate it with respect to time, and use the second order differential equation we derived above for r, to show that energy is in fact constant. (c) Suppose the object falls from radius r_0 with no initial speed. What is its initial energy? (d) Solve the equation expressing conservation of energy to find an expression for r' at any radius r.*

Exercise 7. *How accurate is the table we calculated? If we run the calculations with smaller and smaller values of Δt we expect more and more accurate approximations. Here are some tables for a sample calculation:*

$\Delta t = 1$:

900.0	8293.452	−4.0470	−0.00578

$\Delta = 0.5$:

900.0	8292.197	−4.0480	−0.00578

$\Delta = 0.25$:

900.0	8291.569	−4.0485	−0.00578

$\Delta = 0.125$:

900.0	8291.255	−4.0487	−0.00578

Plot on a graph the calculated value of $r(900)$ versus the time interval Δt (four points altogether). Use the graph and a few calculations to estimate $r(900)$ as accurately as you can.

Summary. A physical system is described by a certain set of variables called its **state variables**. These are chosen to determine the system completely, in the sense that if these are known at a certain time t, then they are also, at least in principle, determined at all other times. For a simple isolated object moving in one dimension, for example one falling vertically in a gravitational field, position $x(t)$ and velocity $v(t)$ are the most convenient state variables. Acceleration is determined in terms of time, position, and velocity (often just position), say $a = f(t, x, v)$, and the evolution of the system over very small intervals of time follows very closely the approximate rules

$$\begin{aligned} a(t) &= f(t, x(t), v(t)) \\ x(t + dt) &= x(t) + v(t)\, dt \\ v(t + dt) &= v(t) + a(t)\, dt \ . \end{aligned}$$

If dt is chosen very small, this will give approximate values of $r(t + n\, dt)$ and $v(t + n\, dt)$ for all $n > 0$ from starting values at time t. The equation for a can also be written

$$x'' = f(t, x, x') \ .$$

This is the most general kind of **differential equation of second order**. An equation of order n is one of the form

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$$

and in this case x, x', etc. are the state variables, n of them. A first order differential equation, for example, will look like

$$x' = f(t, x) \; .$$

The basic result about first order equations is that there is a unique solution satisfying the condition $x(t_0) = x_0$ at a a moment $t = t_0$. This is a slightly more formal way of saying what we have about state variables.

There are, in fact, rather oddly behaved differential equations where this expected behaviour is not what one gets, but they do not not occur for real physical systems. We shall ignore the existence of such equations.