Chapter 6. Second order differential equations

A second order differential equation is of the form

$$y'' = f(t, y, y')$$

where $y = y(t)$. We shall often think of t as parametrizing time, y position. In this case the differential equation asserts that at a given moment the acceleration is a function of time, position, and velocity. Such equations arise often in physical situations, as a mathematical interpretation of Newton's Law $F = ma$. We have already seen this in the example of an object falling vertically to Earth from space.

Second order equations are complicated. We shall begin by looking at examples taken from physics, using this as motivation for the later parts. Before we start, however, we point out one simple basic fact. For first order equations, a solution is completely determined by its value $y(t_0)$ at any given moment t_0, or in other words by a single initial condition. For second order equations we have this analogous assertion:

- *In all normal circumstances, given* t_0 *and values* y_0, v_0 *any second order differential equation* $y'' = f(t, y, y')$ *possesses a unique solution* $y(t)$ *with*

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= v_0 \ . \end{aligned}$$

In other words, we must specify two initial conditions to solve an equation of second order completely. In physics, this is the familiar principle that in order to describe how a system evolves we must specify position and velocity at some one moment, together with some rule for determining acceleration in terms of position and velocity. In other words, the way in which a physical system evolves is determined by (i) a set of state variables which describe the system completely; (ii) a physical law telling how the state variables change in time; and (iii) values of those variables at some one moment. For a simple system the state variables are position and velocity.

1. Weight on a spring

One of the simplest physical systems modelled by second order differential equations is a single weight on a spring which is suspended from a fixed support.

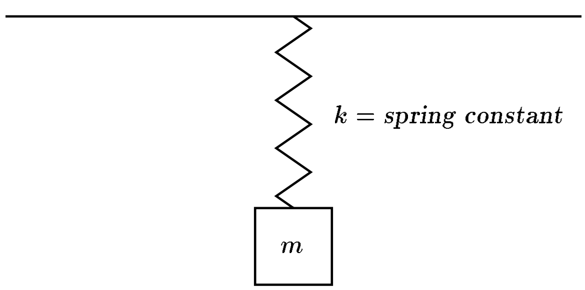


Figure 1.1. *The simplest physical structure.*

We shall assume that the spring obeys Hooke's Law, which asserts that *stress is proportional to strain*. The strain here is vertical force F, the stress is the displacement x of the spring from its relaxed position. The force pulls in the direction opposite to the displacement. Thus

$$F = -kx$$

where k is a constant which depends on the particular spring involved. It is called the **spring constant**.

Suppose the weight has mass m. If it is hanging at rest then the weight mg is exactly balanced by an extension of the spring. If x_{eq} is this equilibrium extension then

$$mg = kx_{\mathrm{eq}}, \quad x_{\mathrm{eq}} = mg/k \ .$$

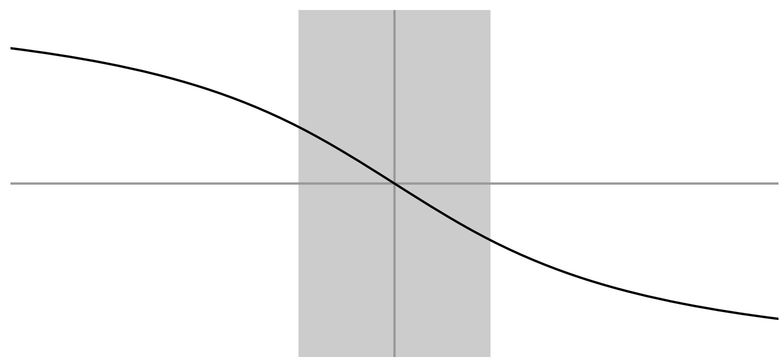
From now on let x be the displacement from this equilibrium position. The total distance of the weight from the support point of the spring is then $\ell + x_{\mathrm{eq}} + x$, if ℓ is the length of the relaxed spring.

There will be force on the weight from weight pulling downwards and either a push or a pull from the extended spring. The total from these effects is $mg - k(x_{\mathrm{eq}} + x) = -kx$.

If there is friction, its exact nature will be rather complicated, but we can approximate it in a plausible way according to the following reasoning: • Friction depends only on rate of change in the system, and its magnitude does not depend on the direction of change. • It acts in opposition to change. From these two facts we can deduce that the force due to friction is of the form $\varphi(v)$ where $v = y'$ is rate of change, $\varphi(-v) = -\varphi(v)$, and $\varphi \le 0$ for $v \ge 0$. The function φ must be **odd**, in other words. If we expand it in a Taylor series around $v = 0$ we will get only odd terms, and for small values of v we will get a good linear approximation

$$\varphi(v) = -cv$$

with $c \ge 0$.



The total amount of force on the weight at displacement x from equilibrium is therefore $-kx - cx'$ (if we are measuring x upwards). According to Newton's Law $F = ma$ we therefore derive the differential equation

$$mx'' = -kx - cx'$$

or

$$mx'' + cx' + kx = 0$$

describing the evolution of the system.

If we allow a support which itself is capable of motion, let $y(t)$ be the vertical displacement of the support from rest at time t. The analysis above leads to a differential equation

$$mx'' + cx' + kx = -ky(t) \ .$$

The term $-ky(t)$ may be thought of as an external force applied to the simpler system.

Exercise 1.1. *A certain spring is extended by 3 cm. when a mass of 40 grams is hung on it. What is the spring constant k?*

Exercise 1.2. *A block of light wood of mass 3 kg. and 20 cm. on a side is floated in deep water. How much shows above water? When it is pushed down into the water, it begins to oscillate. Write down the differential equation describing this oscillation, using the the vertical displacement from equiibrium as the variable y.*

2. Simple electric circuits

Another common physical system modelled by a second order differential equation is a simple electric circuit. Here and later on, we are going to be looking into the behaviour of electric circuits assembled from four different types of components: **resistors**, **inductors**, **capacitors**, and **independent voltage sources**.

If V is the voltage drop across an element and I the current flowing through it, then we have defining relations:

Type of element	Defining property
Resistor	$V = IR$ (Ohm's Law)
Inductor	$V = LI'$
Capacitor	$Q = VC$
	$I = Q'$
	$= V'C$
Independent voltage source	$V = V(t)$ is specified as a function of time

Here R is resistance, L inductance, C capacitance. Also, Q is the charge stored in the capacitor, so that $I = Q'$ is the current effectively flowing through it. The last condition means that the voltage across an 'independent voltage source' is independent of the rest of the circuit.

In this chapter we shall look at the series circuit with one each of these components in a loop.

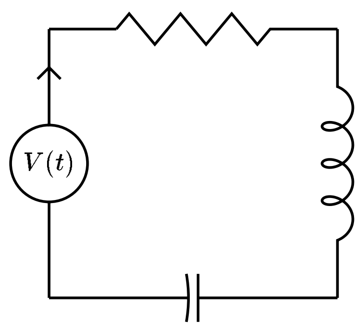


Figure 2.1. *A simple electric circuit.*

Voltage represents potential energy, so the total voltage drop around this circuit must be 0. This means (with the right sign conventions) that

$$LI' + RI + Q/C = V(t) \ .$$

This involves both I and Q, but if we differentiate it we get

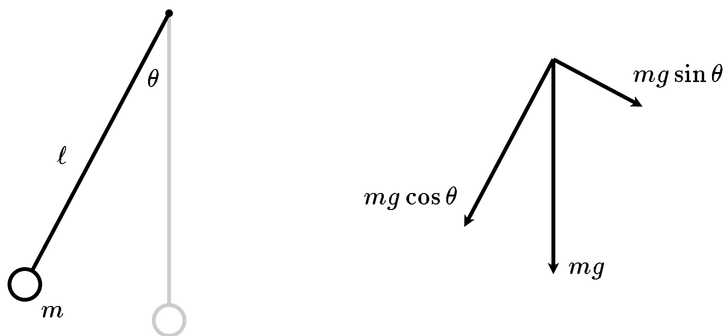
$$LI'' + RI' + I/C = V'(t)$$

where the only unknown function is I. Note that if $L = 0$, we obtain a first order differential equation, which we might have considered in Part 1. Mathematically, such a circuit is equivalent to a cooling object in a variable environment.

Exercise 2.1. *Assume $L = 0$ in the series circuit. Find the general solution with $V(t) = \cos \omega t$. Find the steady state solution.*

3. The pendulum

Here is an example we shall consider several times, sometimes with a few variations. *A pendulum of length ℓ is swinging on a fixed pivot. The mass of its bob is m. The bob is pulled away from the dead vertical position and released. Describe the subsequent motion.*



The first step is to write down what Newton's Second Law asserts in these circumstances. Gravity pulls downward on the bob, but if we assume that the shaft is rigid the component exerted along the shaft will have no effect. The only effect of gravity will therefore be to accelerate the bob in the direction along the periphery of the swing (tangential to the arc of motion). This component is equal to $-mg\sin\theta$ if θ is the angular displacement of the pendulum from equilibrium. The sign is determined by the condition that gravity acts to decrease the angle. If we let y be the linear distance along the arc of the pendulum then $y/\ell = \theta$. Since the force acts to decrease the angular displacement, the tangential force is precisely

$$-mg\,\sin\left(\frac{y}{\ell}\right)\ .$$

There will be another force exerted on the bob, namely that of friction, but we will ignore this for the moment. Newton's Law now tells us that

$$my'' = -mg\sin\left(\frac{y}{\ell}\right)$$

or

$$y'' = -g\,\sin\left(\frac{y}{\ell}\right)\ .$$

Here, again, we therefore obtain a second order differential equation as a direct translation of Newton's Law into mathematics, no more and no less.

The differential equation, as I have mentioned before, tells us how the position and velocity change instantaneously. This change is not very complicated, although because of the $\sin$ term is a bit more complicated than what we have seen before. Monetheless neither Newton's Law nor the differential equation tell us directly how to describe the complete motion explicitly. In other words, the instantaneous chnge is relatively simple, but the long term consequences are not so simple. This is a common feature of physical systems.

Something else in addition to the differential equation is necessary to describe the motion. Let θ_0 be the initial angle of displacement. The bob is released from rest at that displacement, so the initial velocity is 0. In other words we have the **initial conditions** at time $t = 0$

$$y(0) = \ell\theta_0, \quad y'(0) = 0\ .$$

Here, as for all second order differential equations, motion subsequent to any moment is completely determined by position and velocity at that moment.

So far we have ignored friction. For general motion it can be quite complicated, but a reasonable approximation is to let it be proportional to velocity. Anyway, we will usually going to consider only motions of a large massive pendulum which does not move far from the vertical position. In this situation, θ remains small and $\sin\theta$ is approximately the same as θ. Taking this approximation as well as one for friction into account, we get the approximating equation

$$my'' + cy' + ky = 0, \quad k = mg/\ell \ .$$

This is a linear equation, and it is called the **linear approximation** of the original one, for motions in the neighbourhood of the equilibrium position at dead vertical.

Suppose that there is no friction, hence that $c = 0$. The approximating equation is

$$y'' + (g/\ell)y = 0 \ .$$

As we shall see in a moment, the solutions of this are simple periodic functions of period $\sqrt{g/\ell}$, and this is related to the fact that as the amplitude of the pendulum's swing becomes small the period of its swing becomes independent of amplitude.

4. Linear equations

Let me repeat the argument above: if x is small then $\sin x$ is approximately equal to x. Therefore if the pendulum displacement y is small $\sin(y/\ell) \doteq y/\ell$ and the differential equation for the pendulum can be written approximately as

$$my'' + cy' + (mg/\ell)y = 0 \ .$$

This equation is linear in y, and is called a **linear** differential equation. More generally, a linear differential equation (of second order) is one of the form

$$y'' + a(t)y' + b(t)y = f(t) \ .$$

Linear differential equations play an important role in the general theory of differential equations because, as we have just seen for the pendulum

- *Differential equations can be approximated near equilibrium by linear ones.*

The more complicated the physical system, the more important this sort of thing becomes, because they are far more difficult to solve explicitly, and we need all the help we can get.

We have already considered first order linear equations in some detail. For the first order equation

$$y' + ay = f(t)$$

there exists an integral formula

$$y = Ce^{-at} + e^{-at} \int^t e^{as} f(s)\, dt$$

for the general solution. For second order equations we shall see a somewhat similar formula, although it will not turn out to be quite so explicit. The overall structure of the formula, however, will be the same. Recall that the first term in the first order formula is the general solution of the *homogeneous* equation where $f(t)$ vanishes, and the constant C is there because if $y(t)$ is a solution to a first order linear equation then so is $Cy(t)$. Something similar happens for arbitrary homogeneous linear differential equations.

The expression $y'' + a(t)y' + b(t)y$ may be thought of as a kind of **operator**: it transforms a function $y(t)$ into another function of t. For example, the operator $y \mapsto y'$ transforms y into its first derivative. The point here is that this is a **linear operator**. I will explain this formally in just a moment, but first let me give linearity some intuitive content by mentioning that the concept makes sense whenever we can talk about **quantitative input** and **output**. Here the input is y and the output is what we get when we transform y by the operator. Linearity in this context encapsulates two features:

- **Scalability**. *Output is proportional to input.*

Thus, for example, if input is doubled, so is output.

- **Superposition**. *If we add together two inputs, we get out the sum of their respective outputs.*

This means that distinct inputs have an essentially independent effect.

The reason linearity is an important concept is that

- *Many physical systems have essentially linear response near equilibrium.*

We shall see why this is true in the next section.

Linearity in a physical structure, amounts to a generalization of Hooke's Law, which asserts that stress (output) is proportional to strain (input). This is scalability. Superposition is just an extension of this to complicated systems where stress and strain have several components.

In our context, a linear operator is an operator $y \mapsto Ly$ that satisfies these two conditions:

- If c is a constant then $L[cy] = cL[y]$.
- If y_1 and y_2 are two functions then $L[y_1 + y_2] = L[y_1] + L[y_2]$.

These two principles can be combined into one:

- If c_1 and c_2 are constants, y_1 and y_2 functions then $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$.

In this course there are several linear operators:

- **Linear differential operators** which take the function y to a linear combination of its derivatives whose coefficients are functions of t:
$$y \mapsto c_0(t)y + c_1(t)y' + \cdots + c_n(t)y^{(n)} \ .$$
- Any **integral operator**
$$y \mapsto \int_{t_0}^{t} c(s)y(s)\, ds \ .$$

We shall need these simple properties of linear operators:

- *If $L[y_1] = 0$ and $L[y_2] = 0$ and c_1 and c_2 are constants then $L[c_1y_1 + c_2y_2] = 0$.*
- *If $L[y_1] = f_1$ and $L[y_2] = f_2$ and then $L[y_1 + y_2] = f_1 + f_2$.*

These are straightforward and formal to verify.

A linear differential equation
$$y'' + a(t)y' + b(t)y = c(t)$$
is said to be **homogeneous** when the right hand side $c(t)$ is 0. Linearity implies:

- *Linear combinations of solutions to a linear homogeneous differential equation are again solutions of the same equation.*
- *If y_1 and y_2 are solutions of the differential equation*
$$y'' + a(t)y' + b(t)y = c(t)$$
the $y_1 - y_2$ is a solution of the associated homogeneous equation.
- *If we have found one solution y of the differential equation*
$$y'' + a(t)y' + b(t)y = c(t)$$
then every solution of the same equation is of the form $y + z$ where z is a solution of the associated homogeneous equation.

The second assertion is true because

$$L[y_1 - y_2] = L[y_1] - L[y_2] = c(t) - c(t) = 0 \ .$$

The last is just a rephrasing of the second.

An example of the way things work is the basic formula for solutions of a linear first order differential equation. In that formula Ce^{at} is the general solution of the associated homogeneous equation, and the integral is a particular solution of the inhomogeneous one.

There is a practical effect of these considerations on the structure of solutions to a linear equation. We have seen that solutions of an equation of second order depend on two parameters, the position and velocity at any given moment. This and linearity have the consequence that *the solutions of a second order linear homogeneous equation are in some sense a two dimensional vector space*. In the next section we shall do this more precisely in terms of a possible coordinate system on the set of solutions.

5. Why linearity is ubiquitous

Linearity is ubiquitous for a very simple purely mathematical reason. Suppose we are considering a system with one degree of freedom in an equilibrium state, and suppose for the moment that friction is negligible in the system. Suppose also that the physics of the system is not changing in time. Let y be a variable measuring displacement from equilibrium. The equilibrium position is determined by the fact that it must be where potential energy P is a minimum. By elementary calculus, we know that as a function of the variable y the first derivative of P with respect to y vanishes. The Taylor expansion of P near equilibrium then asserts that

$$P(y) = P_0 + y^2 P_2 + \cdots \approx P_0 + \left(\frac{P_2}{2}\right) y^2$$

where the terms left out are of order higher than 2 in y. As in any potential field, the force restoring the system to equilibrium is given by the formula

$$F = -\frac{\partial P}{\partial y} \approx P_2 y \ .$$

Newton's second law then gives us the differential equation

$$my'' = -\frac{\partial P}{\partial y}$$

which for small displacements y is approximately the linear equation

$$my'' = -P_2 y \ .$$

We have already discussed a linear approximation to friction. In summary, the differential equation for the system will be

$$my'' = \varphi(y') - \frac{\partial P}{\partial y}$$

and if y and y' are small it is approximated by the linear equation

$$my'' = -cy' - P_2 y \ .$$

The total energy E stored in the linear system is the sum of kinetic and potential energies

$$E = \frac{mv^2}{2} + \frac{P_2 y^2}{2} + PE_0$$

($v = y'$) and if we differentiate this expression we get from the differential equation

$$\frac{dE}{dt} = mvv' + E_2 yy' = v(my'' + E_2 y) = -cv^2 \ .$$

This says that the energy of the system decreases only through friction losses.

Exercise 5.1. *Write down an expression for the energy of the weight on a spring when the friction coefficient vanishes.*

Exercise 5.2. *Write down an expression for the energy of the pendulum.*

6. Fundamental solutions of a homogeneous linear equation

Suppose we are given a second order differential equation which is linear and homogeneous. Fix a point t_0. If y is any solution of the equation then we can assign to y as coordinates the numbers $y(t_0)$ and $y'(t_0)$, which we know to distinguish y among all solutions. This reinforces the idea that the set of solutions is two dimensional.

Some solutions have simple coordinates. We can find a solution $y_0(t)$ of the differential equation satisfying the simple initial conditions

$$\begin{aligned} y_0(t_0) &= 1 \\ y_0'(t_0) &= 0 \end{aligned}$$

and another y_1 such that

$$\begin{aligned} y_1(t_0) &= 0 \\ y_1'(t_0) &= 1 \ . \end{aligned}$$

If y is any other solution of the differential equation, let $c_0 = y(t_0)$, $c_1 = y'(t_0)$. Then because of linearity $c_0 y_0 + c_1 y_1$ will be a solution of the differential equation, and furthermore by the definition of the y_i it will satisfy the same conditions at t_0 that y does. Therefore this linear combination of y_0 and y_1 must be the same as y, which means that

$$y = c_0 y_0 + c_1 y_1, \quad c_0 = y(t_0), \quad c_1 = y'(t_0) \ .$$

The coordinates of a solution will vary with the choice of t_0. No problem, but keep it in mind. We shall see the solutions y_0 and y_1 again. They are called the **fundamental solutions** of the equation determined by the point t_0. In order to help keep track of the fact that they depend on the particular value of t_0 at hand, I shall write them

$$y_{0,t_0}(t), \quad y_{1,t_0}(t) \ .$$

The index t_0 here is a **parameter** of the solutions, meaning that for each choice of t_0 we get two solutions of the equation with certain properties related to t_0. We shall see some examples soon.