Chapter 3. Periodic functions

Why do lights flicker? For that matter, why do they give off light at all? They are fed by an alternating current which turns into heat because of the electrical resistance of the filament. The heat flow into the filament balances the heat loss from the filament, on the average, and keeps its temperature more or less constant. According to the physics of radiation, an object at high temperature radiates light. But the actual current varies periodically, consequently varying also the heat flow and the light radiation. In order to understand in more detail exactly what is going on, we shall have to understand periodic functions better.

1. Simply periodic functions

An alternating current varies in a simple periodic fashion. How do we express this mathematically?

A **simple** periodic function is one of the form $A\cos\omega t + B\sin\omega t$. In these notes we shall characterize such functions in a uniform manner, and see in particular how to draw their graphs.

The simplest functions of this sort are

$$y = \cos t$$
$$y = \sin t\ .$$

They both have period 2π, and their graphs look like this:

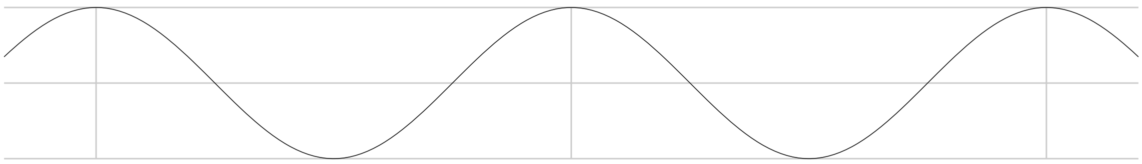


Figure 1.1. *The graph of* $y = \cos t$.

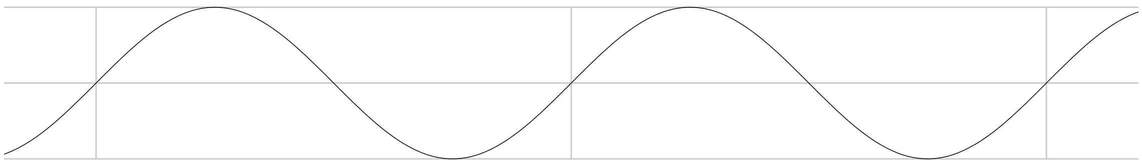


Figure 1.2. *The graph of* $y = \sin t$.

These two graphs are in fact essentially the same—they differ only in that the graph of $\sin t$ is obtained from that of $\cos t$ by shifting it to the right by an interval of $\pi/2$, which is one quarter of a period. Now if we are given a function $f(t)$ and shift its graph to the right by an interval of h, we obtain the graph of $f(t-h)$ (for example, at $t = h$ the shifted graph will have the same height as that of f at 0). Therefore the shift in the graphs of $\cos t$ and $\sin t$ is a consequence of the trigonometrical identity

$$\sin t = \cos(t - \pi/2)\ .$$

Of course it is also possible to write

$$\cos t = \sin(t + \pi/2)\ .$$

However, it is conventional to use the function $\cos t$ as the basic periodic function, and express all others in terms of it. Thus we shall say that $\cos t$ has a **phase shift** of 0 and that $\sin t$ has a phase shift of $\pi/2$.

Now consider the functions

$$y = \cos 2t$$
$$y = \sin 2t$$

The graphs of these functions are obtained from those of $\cos t$ and $\sin t$ by compressing them along the x-axis by a factor of 2. The period of the new functions is equal to π, or half that of the original ones, their **frequency** is twice that of the originals.

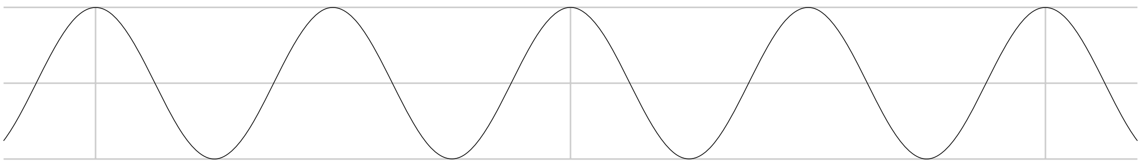


Figure 1.3. *The graph of* $y = \cos 2t$.

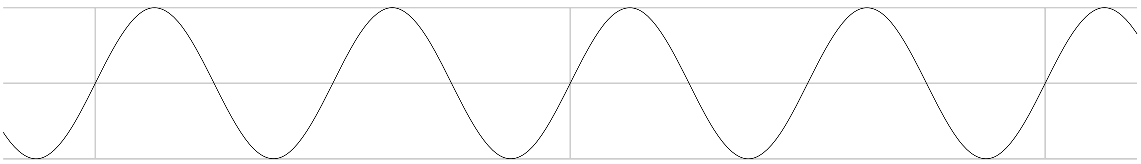


Figure 1.4. *The graph of* $y = \sin 2t$.

The functions

$$\cos \omega t, \quad \sin \omega t$$

will have **period** $T = 2\pi/\omega$ since, for example

$$\cos \omega(t + T) = \cos \omega(t + 2\pi/\omega) = \cos(\omega t + 2\pi) = \cos \omega t \ .$$

In one unit of time it will cover $1/T$ cycles, so that it is also said to have **true frequency**

$$1/T = \omega/2\pi$$

but it is awkward to incorporate the factor of 2π regularly, and it is usually more convenient to refer to the **radian frequency** ω, since in one period the variable will pass through 2π radians.

It is still true that the graph for one of the functions $\cos 2t$ and $\sin 2t$ is obtained from the other by a shift. More precisely, that of $\sin 2t$ is obtained from that of $\cos 2t$ by a shift of $\pi/4$, or half what the previous shift was, since

$$\sin 2t = \cos 2(t - \pi/4)$$

But we can also say that the shift still amounts to a **quarter cycle**, and rewrite the equation as

$$\sin 2t = \cos(2t - \pi/2) \ .$$

It turns out that it is not usually the absolute value of the shift which is important, but rather the *amount of shift relative to the length of a cycle*. This is called the **shift in phase**, since in one cycle the phase of a periodic function varies from 0 to 2π. The **phase shift** of the function

$$\cos(\omega t - \theta) = \cos \omega(t - \theta/\omega)$$

is therefore defined to be θ. It represents the proportion $\theta/2\pi$ of a single cycle. The true, or absolute, time shift θ/ω plays only a small role in dealing with periodic functions.

Now consider the function $y = 2\cos t$. Its relation to $\cos t$ is very simple, since it just oscillates with a greater **amplitude**.

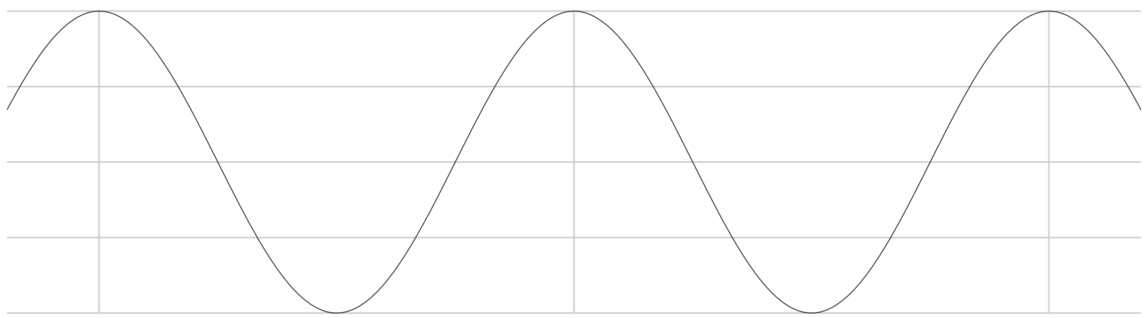


Figure 1.5. *The graph of* $y = 2\cos t$.

To summarize:

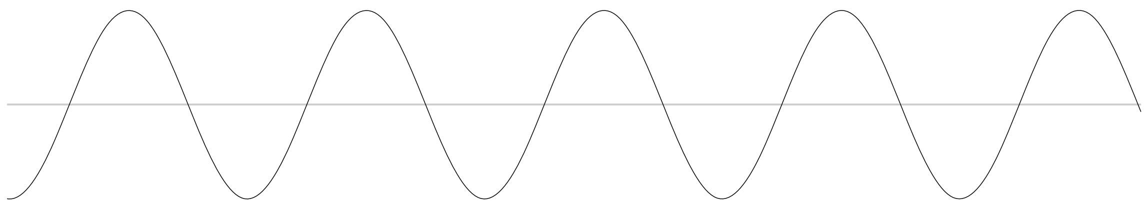
- *A* **simply periodic function** *is one of the form*

$$A\cos(\omega t - \theta)$$

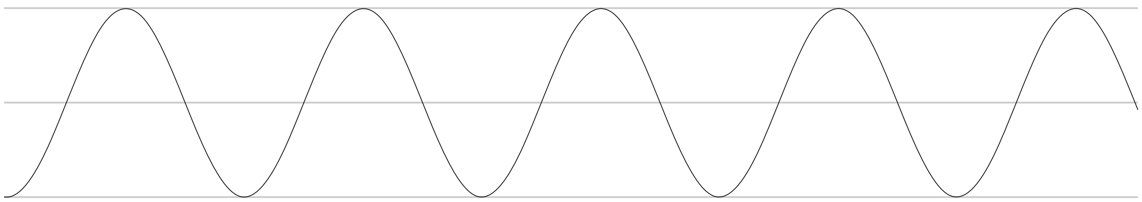
It is said to have **amplitude** A, **radian frequency** ω, **phase shift** θ.

Such a function can be graphed in a very simple but perhaps slightly unorthodox sequence of steps. Consider the curve $y = 1.25\cos(2x - \pi/4)$, for example.

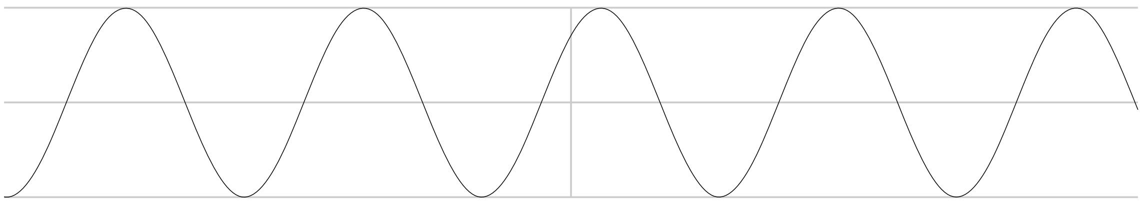
(1) Draw the x-axis and then the graph of $y = \cos t$. This is the unorthodox part of this construction—we draw the graph first and then assemble a framework around it in order to interpret it.



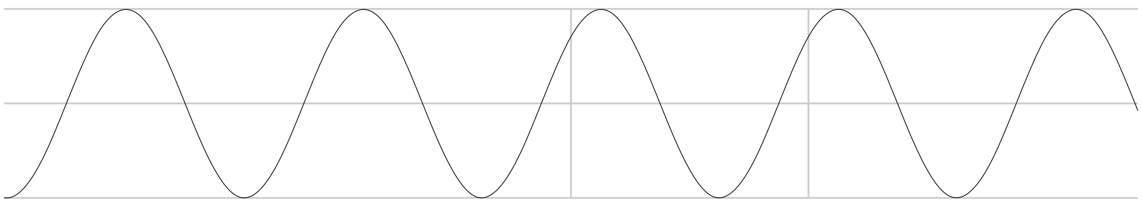
(2) Add the horizontal lines $y = \pm A$ to establish the y-scale.



(3) Add the y-axis to match the phase shift, establishing the origin of the x-axis. Here the shift is $1/8$ of a cycle.



(4) Finally establish the x-scale by laying out one period.



The final result we need about simple periodic functions is a generalization of the relationship between $\cos x$ and $\sin x$.

- *Any linear combination*

$$A\cos\omega x + B\sin\omega x$$

  is a simple periodic function.

In other words, it can therefore be expressed in an essentially unique manner

$$A\cos\omega x + B\sin\omega x = C\cos\omega(x-\theta)$$

We must find C and θ. If we write out

$$\cos(\omega x-\theta) = \cos\omega x\cos\theta + \sin\omega x\sin\theta$$

we see that we must have

$$C\cos\theta = A$$
$$C\sin\theta = B$$

If we square these equations and add then since $\cos^2 + \sin^2 = 1$ we get

$$C^2 = A^2 + B^2, \quad C = \sqrt{A^2 + B^2}$$

and once we calculate C we can calculate θ according to the equations

$$\cos\theta = A/C$$
$$\sin\theta = B/C$$

Example. The amplitude of

$$\cos t + \sin t$$

is $C = \sqrt{2}$ and its phase shift is $\pi/4$.

Exercise 1.1. *Sketch carefully the graph of* $y = \cos t - \sin t$.

Exercise 1.2. *Find the amplitude and phase shift of* $3\cos(t/3) - \sin(t/3)$. *Sketch its graph carefully.*

2. Relations with complex numbers

A complex number z can be written as

$$z = a + ib$$

but also as

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

which means that it has polar coordinates r and θ. Sometimes r is called the amplitude of z and θ its phase. The two expressions are related according to the formula

$$r = \sqrt{a^2 + b^2}$$
$$\cos\theta = a/r$$
$$\sin\theta = b/r\ .$$

The reason these formulas look so much like the previous ones is this: the function $a\cos(\omega t - \theta)$ may be thought of as the real part of the complex exponential

$$re^{i(\omega t - \theta)} = re^{i\omega t}e^{-i\theta} = r(\cos\omega t + i\sin\omega t)(\cos\theta - i\sin\theta)$$

whose real part is

$$r\cos\omega t\cos\theta + r\sin\omega t\sin\theta\ .$$

But we can also think of

$$a\cos\omega t + b\sin\omega t$$

as the real part of

$$e^{i\omega t}(a - ib)$$

so we can write

$$a + ib = re^{i\theta}, \quad a - ib = re^{-i\theta}$$

and

$$a\cos\omega t + b\sin\omega t$$

as the real part of

$$re^{i(\omega t-\theta)}\ .$$

We shall see later that this way of using complex numbers can simplify the solution of some differential equations.

/RUN def

ion Transients and steady state Now that we understand periodic functions, we can investigate objects cooling in an environment whose temperature oscillates with time.

Let's start off by looking at the case where the environment temperature is an arbitrary function of t. We are looking at the differential equation

$$\theta' = -k\theta + k\theta_{\rm env}(t)$$

which we can rewrite as

$$\theta' + k\theta = k\theta_{\rm env}(t)\ .$$

If $\theta(0) = \theta_0$, we can apply the general formula to get

$$\theta(t) = \theta_0\, e^{-kt} + e^{-kt}\int_0^t k\theta_{\rm env}(s)e^{ks}\, ds\ .$$

If $\theta_{\rm env}$ is a constant Θ_0, then

$$\int_0^t k\Theta_0 e^{ks}\, ds = \left[\frac{k\theta_{\rm env}e^{ks}}{k}\right]_0^t = \Theta_0(e^{kt}-1)$$

and we get the previous formula

$$\theta = \theta_0 e^{-kt} + e^{-kt}\Theta_0(e^{kt}-1) = \Theta_0 + e^{-kt}(\theta_0-\Theta_0)\ .$$

A more interesting case is when $\theta_{\rm env}$ varies periodically with t. Suppose $\theta_{\rm env} = \Theta_0\cos\omega t$. Then we get

$$\theta = \theta_0 e^{-kt} + k\Theta_0 e^{-kt}\int_0^t e^{ks}\cos\omega s\, ds\ .$$

To evaluate the integral we set $\cos\omega s$ equal to the real part of $e^{i\omega s}$. We have

$$\begin{aligned}\int_0^t e^{ks}e^{i\omega s}\, ds &= \int_0^t e^{(k+i\omega)s}\, ds\\ &= \left[\frac{e^{(k+i\omega)s}}{k+i\omega}\right]_0^t\\ &= \frac{e^{(k+i\omega)t}-1}{k+i\omega}\\ &= \frac{k-i\omega}{k-i\omega}\ \frac{e^{kt}\cos\omega t + ie^{kt}\sin\omega t - 1}{k+i\omega}\ .\end{aligned}$$

This gives us for the solution

$$\theta = \Theta_0\left(\frac{k^2\cos\omega t + k\omega\sin\omega t - k^2e^{-kt}}{k^2+\omega^2}\right) + \theta_0 e^{-kt}$$

which can be expressed as

$$\Theta_0\left(\frac{\cos\omega t + (\omega/k)\sin\omega t}{1+(\omega/k)^2}\right) + e^{-kt}\left(\theta_0 - \frac{\Theta_0}{1+(\omega/k)^2}\right)\ .$$

This formula is perhaps difficult to understand. The best way to understand it is first to write

$$k + i\omega = k\big(1+i(\omega/k)\big) = kR_\omega e^{i\theta_\omega}$$

where

$$R_\omega = \sqrt{1+(\omega/k)^2}$$

is the amplitude and θ_ω is the argument of the complex number $k+i\omega = \big(1+i(\omega/k)\big)$. We then have

- *The solution to*

$$\theta' = -k(\theta - \Theta_0 \cos \omega t), \quad \theta(0) = \theta_0$$

is

$$\theta = \frac{\Theta_0}{R_\omega} \cos(\omega t - \theta_\omega) + e^{-kt} \left(\theta_0 - \frac{\Theta_0}{R_\omega^2} \right)$$

where

$$1 + i(\omega/k) = R_\omega e^{i\theta_\omega} \ .$$

The temperature $\theta(t)$ has two components. One oscillates in time with the surrounding temperature θ_{env}, while the other decreases at an exponential rate. For large values of t the second will be extremely small. It is called the **transient component**. The second remains about the same size, on the average, and is called the **steady state component**. It might be thought of as a kind of oscillating equilibrium. If $\omega = 0$ we recover again the approach to the ordinary equilibrium. If $\omega > 0$ the number R_ω is always greater than 1, so the amplitude of the oscillating component is always less than the amplitude of the oscillation of the surrounding temperature. Furthermore, the angle θ_ω will always lie between 0 and $\pi/2$, so the object's temperature will lag behind that of the surrounding temperature by as much as one quarter of a cycle. The rough description of this behaviour is that *the temperature of the object tracks that of its environment, but rather sluggishly*—and more sluggishly for higher frequencies of oscillation in the environment.

Exercise 2.1. *Write down the differential equation satisfied by a small object in a room with oscillating temperature $\theta_{\mathrm{env}}(t) = \cos t$, and a relaxation time of $\tau = 1$. Write down a formula for $\theta(t)$ if $\theta(0) = 100°$. Write down and then graph the transient and steady state components.*

Exercise 2.2. *If $\theta_{\mathrm{env}}(t) = 24 + \cos t$, the relaxation time is $\tau = 2$, and the initial temperature is 40, graph both $\theta_{\mathrm{env}}(t)$ and $\theta(t)$ as functions of time. Graph the amplitude of the steady state response in these circumstances to $\theta_{\mathrm{env}}(t) = 24 + \cos \omega t$ as a function of ω.*

Exercise 2.3. *Find a formula for the solution of Newton's cooling equation when the room temperature is t (a constantly rising room temperature), initial temperature θ_0. What is $\theta(t)$?*

We now look again at the flickering light, which we will discuss in a qualitative way. The current will be a purely periodic function of frequency 60 cycles, hence of the form

$$I = I_0 \cos(2\pi t/60 - \alpha) = I_0 \cos(\omega t - \alpha)$$

where $\omega = 2\pi/60$ is the frequency and α is the phase lag. We may assume α to be 0 by shifting our clocks to match it. A current I flowing into a resistor R corresponds to a rate of heat flow into the resistor of size I^2R. In other words, we are now looking at a situation slightly different from what we have seen previously, where an object is cooling off in an environment, but is also being heated directly at the same time. The rate of temperature change due to the heat flow will be proportional to the heat flow, and for a small object will be

$$\frac{h(t)}{\rho s}$$

where $h(t)$ is the rate of heat flow, s the **specific heat** of the substance from which the object is made, and ρ is the density of this substance. We arrive at a new version of Newton's Law in the presence of heat flow:

- *A heated object cooling off in an environment at temperature satisfies*

$$\theta' = -k(\theta - \theta_{\mathrm{env}}) + \frac{h(t)}{\rho s}$$

where $h(t)$ is the rate of heating.

In our case, with an alternating current, we have

$$I^2R = I_0^2 R\cos^2\omega t\ .$$

This is not quite a simply periodic function. Instead, we need a little trigonometry to write

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x\ .$$

Adjusting constants, we arrive at a differential equation of the form

$$\theta' = -k\theta + a + b\cos 2\omega t$$

for suitable constants a and b. The first thing this tells us is that the frequency of variation of heating is not 60 cycles, but rather 120. Intuitively, this is because the heating is independent of the direction of the current, which reverses itself every half-cycle. Notice also that there is a kind of background effect caused by the average power produced in one cycle.

3. Discontinuous input

Another situation that might arise is one where the environment changes temperature discontinuously. For example, suppose we start a cup of coffee cooling, but then put it in an oven at $50°$ for awhile to reheat it. In this case we are looking at an equation something like

$$\theta' = -k(\theta - \theta_{\mathrm{env}}) + \theta_{\mathrm{env}}(t)$$

where

$$\theta_{\mathrm{env}} = \begin{cases} 20 & 0 \le t < 10 \\ 50 & 10 \le t < 20 \\ 20 & 20 \le t \end{cases}$$

There are many ways to deal with this sort of problem. The simplest conceptually, and the only one I mention here, is to realize that when we put it into the oven we are just restarting the whole process all over again at that moment. So first we solve the differential equation & initial conditions in the interval $0 \le t < 10$. We get some temperature function $\theta(t)$ for this range, the final value being θ_{10}. Then we solve the equation with new initial conditions $\theta(10) = \theta_{10}$ in the range $[10, 20]$, getting a final value θ_{20}. We finally solve the equation with initial conditions $\theta(20) = \theta_{20}$.

Exercise 3.1. *Suppose $\theta(0) = 100°$, $\tau = 20$, and the room temperature is*

$$\theta_{\mathrm{env}}(t) = \begin{cases} 20 & \textit{for} \quad 0 \le t < 10 \\ 0 & \textit{for} \quad 10 \le t \end{cases}$$

Find $\theta(t)$.

Exercise 3.2. *A cup of coffee initially at $100°$ cools to $40°$ after 10 minutes in a $20°$ room, and is then put in an oven at $120°$ for 5 minutes. What is its temperature then?*

4. Linearity

The solution to

$$y' = ay + b(t)$$

is

$$y = Ce^{at} + e^{at}\int^t e^{-as} b(s)\, ds\ .$$

We want to say something about the general structure of this formula.

If $b \equiv 0$ then the differential equation is

$$y' = ay$$

and its solution is

$$y = Ce^{at}$$

where C is arbitrary. This differential equation is **linear** since it is a linear function of y, and **homogeneous** since the term in y is of a single degree one. The most general equation of this type is

$$y' = a(t)y$$

This equation has the property that *any scalar multiple of a solution by a constant is also a solution.* Later on we shall generalize this **linearity principle**. The solution of the general equation—with $b(t)$ not necessarily equal to 0—has the form

$$\text{general solution to homogeneous part} + \text{a part which depends linearly on } b(t) \ .$$

This also will generalize.

Exercise 4.1. *Find the general solution of*

$$y' = -y + 1 + e^{-t} + \cos t \ .$$

Exercise 4.2. *Suppose we are looking at a cup of coffee initially at 100° where the relaxation time is 20 minutes, in an environment whose temperature is fluctuating at a simple frequency of period 10 minutes according to the formula*

$$\theta_{\mathrm{env}}(t) = 5\cos 2\pi(t/10) \ .$$

What is the temperature of the coffee after 20 minutes?