Chapter 9. Periodic input

1. Transients and steady state behaviour

Simple physical systems, such as a weight on a spring in ideal conditions where Hooke's Law is valid and friction forces are simple, are modeled by the second order differential equation

$$my'' + cy' + ky = F(t)$$

where

$$\begin{aligned} m &= \text{mass} \\ c &= \text{friction constant} \\ k &= \text{proportionality constant of the restoring force} \\ F(t) &= \text{force acting on the system.} \end{aligned}$$

In practice the friction constant c is nearly always positive, so that the system does have friction losses (anything else involves some way of putting energy into the system so as to simulate negative friction).

It is most important to understand what happens when the input force $F(t)$ is a simple periodic function. Then, up to a phase shift, we have

$$F(t) = \cos \omega t$$

for some input frequency ω. We know that the way to solve this is to think of $\cos \omega t$ as the real part of $e^{i\omega t}$, so we first solve

$$my'' + cy' + ky = e^{i\omega t}$$

and then look at the real parts of its solution.

To solve this equation, we first solve the homogeneous equation. This means finding the roots of the characteristic polynomial

$$m\lambda^2 + c\lambda + k = 0$$

We get roots

$$\begin{aligned} \lambda &= \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \\ &= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{aligned}$$

Let Δ be the quantity under the square root sign:

$$\Delta = \left(\frac{c}{2m}\right)^2 - \frac{k}{m}$$

There are three possible cases:

The first is when $\Delta > 0$. The roots λ_1, λ_2 are real and distinct. The solutions of the homogeneous equation are linear combinations of

$$e^{\lambda_1 t}, \quad e^{\lambda_2 t}$$

- *In case $\Delta > 0$, both roots are real and strictly negative.*

The root

$$-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

is the sum of two negative numbers. The other one

$$-\frac{c}{2m}+\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}$$

is negative because we are adding to $c/2m$ the square root of a number $(c/2m)^2-(k/m)$ which is less than $c/2m$. As a consequence, in this case both solutions decay exponentially as $t\to\infty$.

The second is when $\Delta=0$. Then there is a single root

$$\lambda=-\frac{c}{2m}$$

which is negative. The solutions are

$$e^{\lambda t},\quad te^{\lambda t}$$

- *In case $\Delta=0$ both solutions also decay at an essentially exponential rate.*

The third is when $\Delta<0$. Then we set

$$a=-\frac{c}{2m},\quad b=\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}$$

and the solutions are

$$e^{at}\cos bt,\quad e^{at}\sin bt\ .$$

Since $a<0$:

- *In case $\Delta<0$ both solutions oscillate with exponentially decaying amplitude.*

Thus the solutions to the homogeneous equation can behave in various ways, but in physical systems with friction they all have at least one thing in common, and for practical purposes this is often all that we need to know:

- *In all three cases, the solutions to the homogeneous equation decay to 0 as $t\to\infty$ as long as the friction term is not zero.*

In other words, all solutions to the homogeneous equation have only temporary effect—they are said to be **transient**.

For dealing with the inhomogeneous part we set

$$y=\frac{e^{i\omega t}}{m(i\omega)^2+c(i\omega)+k}=\frac{e^{i\omega t}}{Z}$$

where

$$Z=(k-m\omega^2)+ic\omega\ .$$

Explicitly it is equal to

$$\frac{(k-m\omega^2-ic\omega)(\cos\omega t+i\sin\omega t)}{(k-m\omega^2)^2+c^2\omega^2}=\frac{[(k-m\omega^2)\cos(\omega t)-c\omega\sin(\omega t)]+i[c\omega\cos(\omega t)+(k-m\omega^2)\sin(\omega t)]}{(k-m\omega^2)^2+c^2\omega^2}$$

This is a messy expression and, frankly, not all that useful. It is far more useful to think of the solution in terms of amplitude and phase shift.

The complex number

$$Z=(k-m\omega^2)+ic\omega$$

can be expressed as

$$Z = Re^{i\theta}$$

where

$$R = \sqrt{(k - m\omega^2)^2 + c^2\omega^2}$$

and θ is the angular coordinate of Z if it is plotted as $(k - m\omega^2, c\omega)$ on the (x, y) plane. For calculation we need to know

$$\cos\theta = (k - m\omega^2)/R$$
$$\sin\theta = c\omega/R$$

where R has already been calculated.

Thus we can write the solution as

$$\frac{e^{i\omega t}}{Re^{i\theta}} = \frac{1}{R}\, e^{i(\omega t - \theta)}$$

and its real part is

$$\frac{1}{R}\,\cos(\omega t - \theta)\ .$$

The general solution to the differential equation is a sum of the solution to the homogeneous equation plus the part we have just seen. The component which solves the homogeneous decays as $t \to \infty$ and is called the transient component. The other term is a persistent oscillation with the same frequency as the input force. It is called the **steady state** component. In practice, the transient component decays so rapidly that one has to work hard to realize it exists, and in many situations only the steady-state component is of importance.

To summarize and extend a bit:

- *The steady state solution to a second order differential equation*

  $$my'' + cy' + ky = f(t)$$

  where $f(t)$ is any simple period function of frequency ω is of the form

  $$F(t) = \frac{1}{R} f(t - \theta)$$

  where

  $$Z = (k - m\omega^2) + ic\omega = Re^{i\theta}\ .$$

In other words, the shape of the output is the same as the input, but it is magnified (or shrunk) in size and shifted in phase.

We have seen this for the special case $f(t) = \cos\omega t$. If f is an arbitrary simple periodic function of frequency ω then it can be written as

$$a\cos\omega t + b\sin\omega t = A\cos(\omega t - \alpha) = A\cos\omega(t - \alpha/\omega)\ .$$

where

$$A = \sqrt{a^2 + b^2}$$
$$\cos\alpha = \frac{a}{A}$$
$$\sin\alpha = \frac{b}{A}$$

The principle of time-invariance implies that a shift in time on the right hand side of an inhomogeneous equation implies a shift in time of the solution as well, and this is just what the assertion amounts to.

In electric circuits the complex number Z is very closely related to what is called the **complex impedance**.

What is most important in most physical problems is to understand how the amplitude $1/R$ of the steady-state solution depends on the input frequency. This is roughly because its magnitude measures how energy is being swallowed up by friction in the process. Mathematically, we want to find the graph of the amplitude versus ω, among other things. I will show some examples in a moment, but roughly speaking the qualitative nature of this graph depends on how large the friction constant c is compared to k and m.

Exercise 1.1. *Find and graph carefully the steady state solution of*

$$y'' + 2y' + 2y = \cos t \ .$$

Find also the relaxation time of its transient.

2. Resonance

In many physical systems, things have been arranged so as to minimize friction. The price to be paid for this is that the system is then subject to **resonance phenomena**. This is sometimes a good thing, sometimes not.

If a physical system is governed by the differential equation

$$my'' + cy' + ky = \cos \omega t$$

then the steady state solution is

$$\frac{\cos(\omega t - \theta)}{R}$$

where

$$Re^{i\theta} = (k - m\omega^2) + ic\omega \ .$$

The formula for the amplitude of the steady state solution is therefore

$$A = \frac{1}{R}$$

where

$$R = \sqrt{(k - m\omega^2)^2 + c^2\omega^2} \ .$$

We are interested in understanding how amplitude varies as a function of ω. Here is a sample:

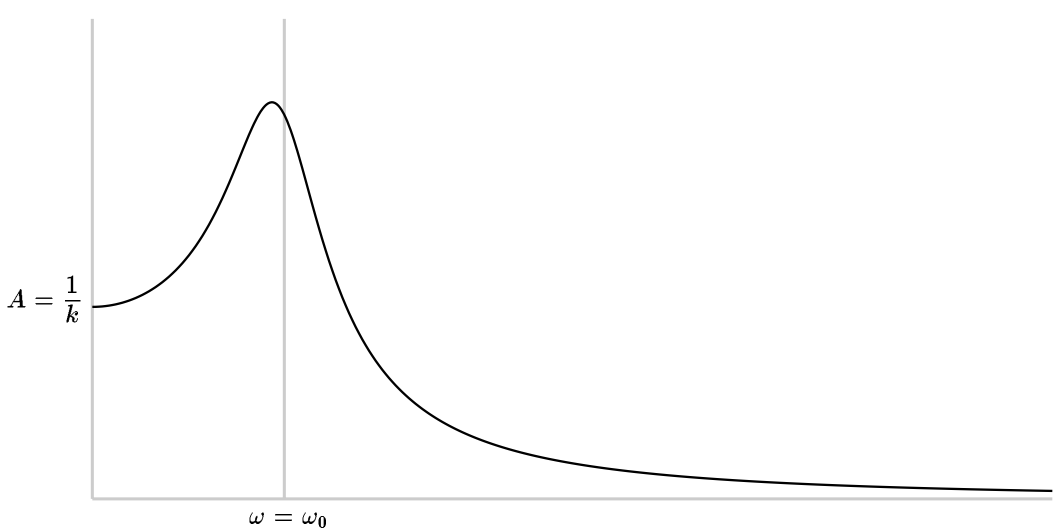


Figure 2.1. *The graph of A versus ω for a system with relatively little friction.*

This is typical behaviour for many physical systems. The main point in understanding how the amplitude depends on ω is to know when and how it achieves its maximum value.

The amplitude is a maximum when R is a minimum. The quantity R is a minimum when the quantity under the square root sign is a minimum. We can determine at least where it achieves minimum and maximum values by the standard technique of calculus. The derivative of

$$(k - m\omega^2)^2 + c^2\omega^2$$

with respect to ω is

$$2(-2m\omega)(k - m\omega^2) + 2\omega c^2 \ .$$

It is always vanishes when $\omega = 0$. To find where else it vanishes we can factor out 2ω and solve:

$$\begin{aligned}
-2m(k - m\omega^2) + c^2 &= 0 \\
k - m\omega^2 &= \frac{c^2}{2m} \\
m\omega^2 &= k - \frac{c^2}{2m} \\
\omega^2 &= \frac{k}{m} - \frac{c^2}{2m^2} \\
\omega &= \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}
\end{aligned}$$

This makes sense only if

$$\frac{k}{m} - \frac{c^2}{2m^2} \ge 0, \quad \frac{c^2}{2m^2} \le \frac{k}{m}, \quad c \le \sqrt{2km} \ .$$

The value of ω where the amplitude takes its maximum value is called the **resonant frequency** unless it is zero.

Before we go further, I want to simplify the expression for A. At first sight, its dependence on m, c, and k is rather complicated. We can make it look better. We can factor out a factor $1/k$, then introduce the frequency $\omega_0 = \sqrt{k/m}$, the resonant frequency for the associated frictionless system:

$$\begin{aligned}
A &= \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \\
&= \frac{1}{k}\,\frac{1}{\sqrt{(1 - m\omega^2/k)^2 + c^2\omega^2/k^2}} \\
&= \frac{1}{k}\,\frac{1}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + (c/\sqrt{km})^2(\omega/\omega_0)^2}} \\
&= \frac{1}{k}\,\frac{1}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + \gamma^2(\omega/\omega_0)^2}} \\
&= \frac{1}{k}\,\frac{1}{\sqrt{(1 - \overline{\omega}^2)^2 + \gamma^2\overline{\omega}^2}}
\end{aligned}$$

where

$$\gamma = \frac{c}{\sqrt{km}}$$

is a normalized friction constant and

$$\overline{\omega} = \frac{\omega}{\omega_0}$$

is a normalized frequency.

Here is a graph of amplitude against input frequency for a typical family of systems with fixed k and m but varying c.

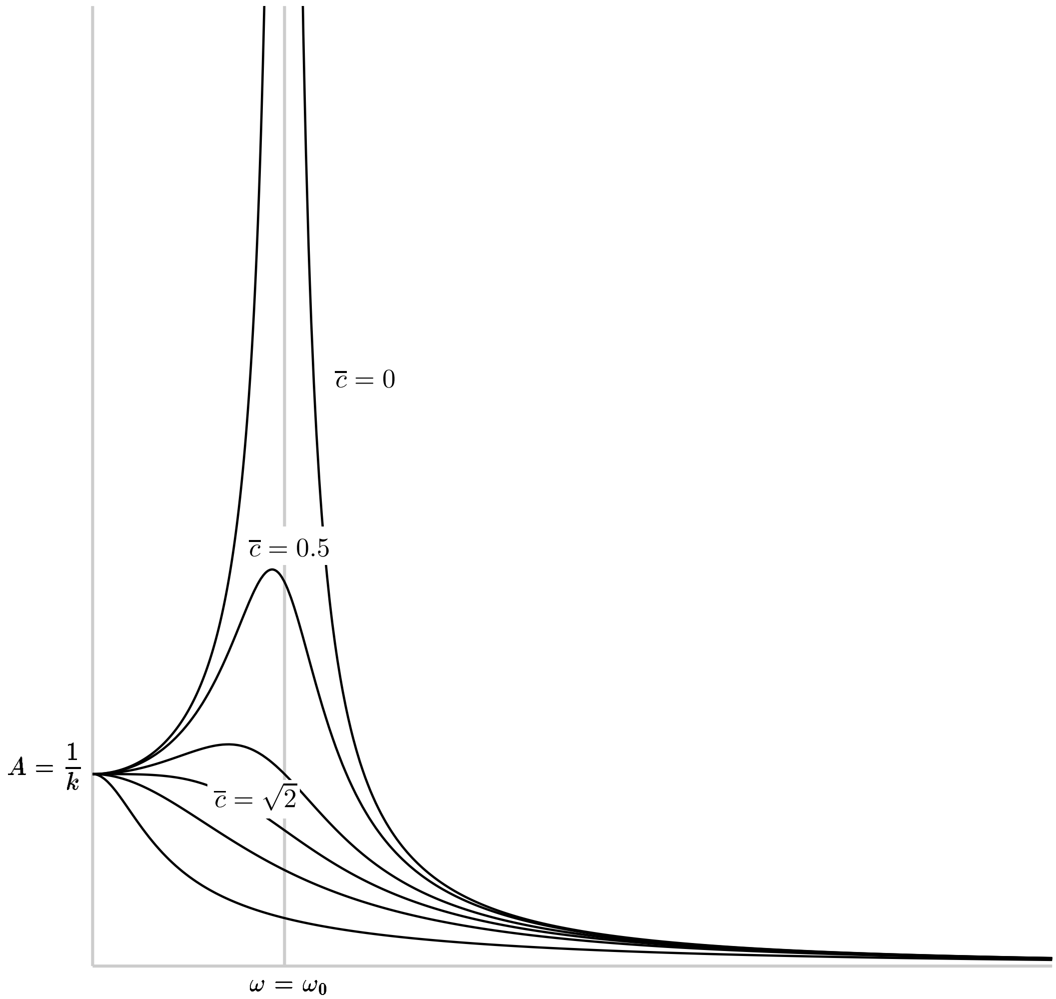


Figure 2.2. *Graphs of amplitude versus frequency for a range of values of γ.*

The highest curve is for $\gamma = 0$, with no friction. Here the amplitude becomes infinite at resonance, which means that in fact there is no steady state frequency. For systems with a low value of γ the peak in the graph is still quite sharp, but as γ approaches $\sqrt{2}$ the peak diminishes and moves left towards $\omega = 0$. For high values of γ the amplitude tapers off directly from the amplitude at $\omega = 0$.

Notice that in the region where resonance is at all strong the resonant frequency is always close to ω_0. This means in efferct that the frictionless system is a good approximation to one with slight friction.

The phase shift also varies with input frequency. The best way to picture how it varies is to plot the complex numbers Z as ω varies from 0 to ∞. In normalized terms we have

$$Z = k\left((1 - \overline{\omega}^2) + i\gamma\overline{\omega}\right) .$$

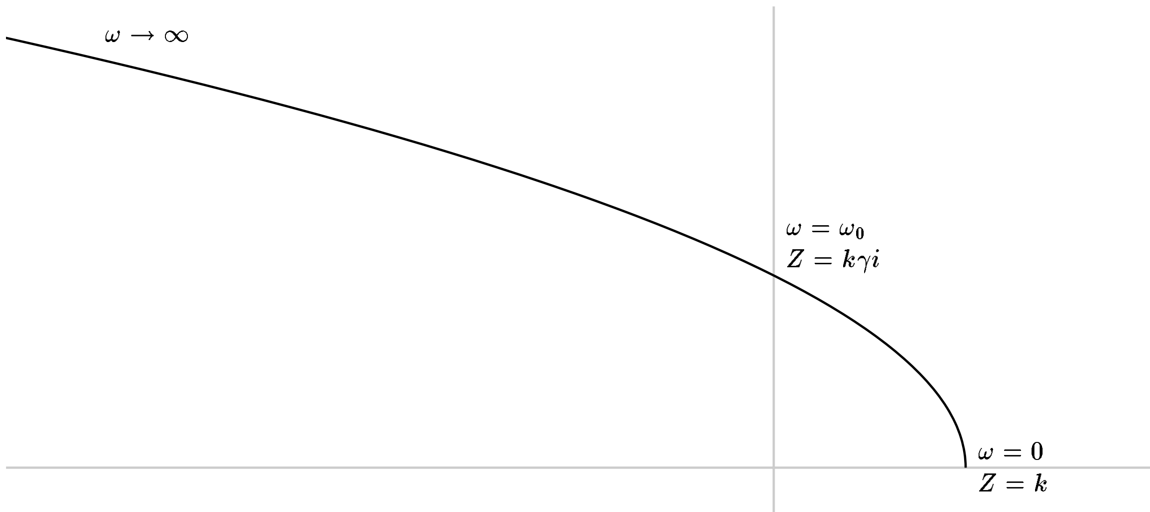


Figure 2.3. *Plot of Z as $\overline{\omega}$ ranges from 0 to ∞.*

Here is again a family of graphs for various values of γ. For $\gamma = 0$ there is a sharp transition from $\theta = 0$ to $\theta = \pi$, and for high values the transition is slow and steady.

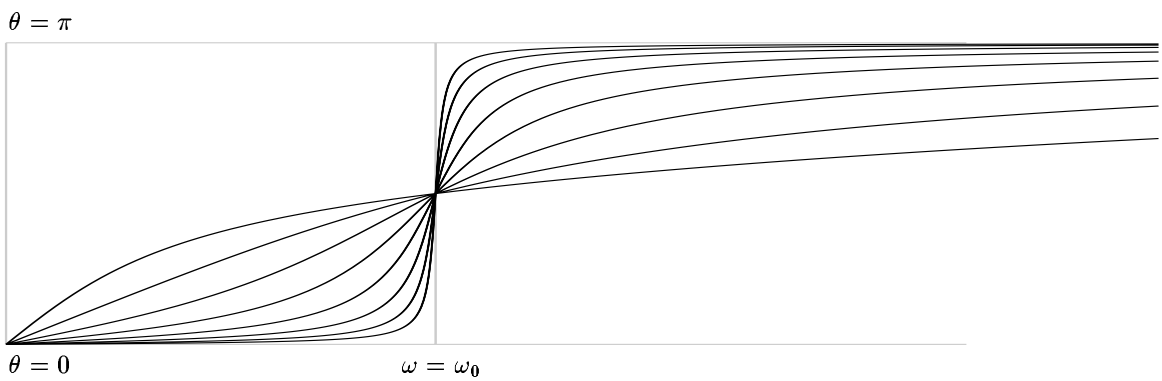
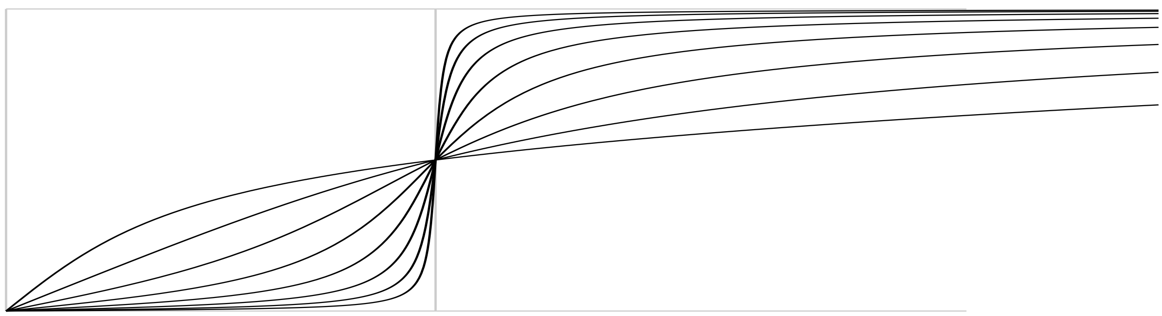


Figure 2.4. *Phase shift θ versus ω.*

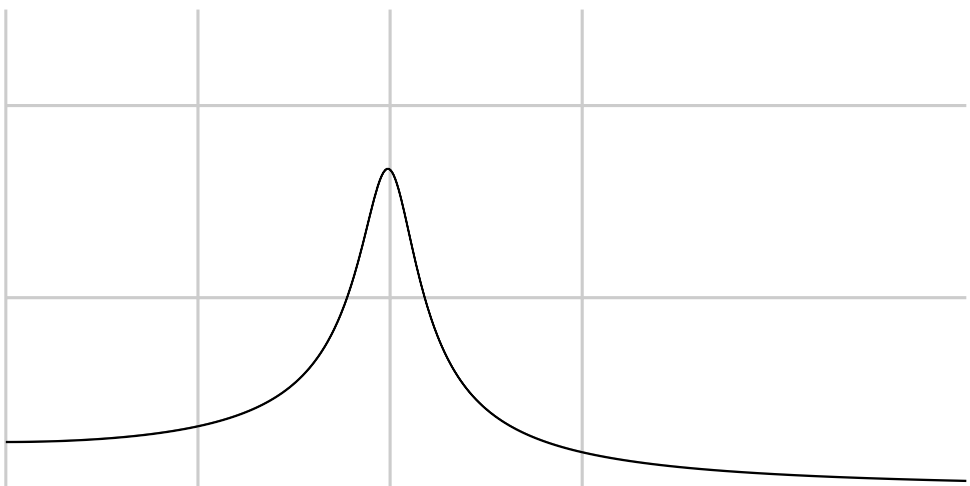
Exercise 2.1. *For the equation $y'' + 2y' + y = \cos \omega t$ graph both amplitude and phase shift versus ω.*

Exercise 2.2. *To each of the plots of phase shift in this figure, find the value of the shift (starting with the bottom and going up):*



Exercise 2.3. *Use the formula for amplitude in terms of γ and $\overline{\omega}$ to find a formula for the maximum amplitude when resonance exists.*

Exercise 2.4. *The following is the resonance graph for a certain differential equation $my'' + cy' + ky = \cos\omega t$. Find m, k, c, at least approximately.*



3. The approach to resonance

We know that the general solution to

$$y'' + \omega_0^2 y = \cos\omega t$$

is one of two forms:

$$c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \begin{cases} \dfrac{\cos\omega t - \cos\omega_0 t}{-\omega^2 - \omega_0^2} & \omega \neq \omega_0 \\[2ex] \dfrac{t\sin\omega t}{2\omega} & \omega = \omega_0 \end{cases}$$

From this it is a simple calculation to get the following:

- *If $\omega \neq \omega_0$ then the solution to*

$$y'' + \omega_0^2 y = \cos\omega t, \quad y(0) = y'(0) = 0$$

is

$$\frac{\cos\omega t - \cos\omega_0 t}{-\omega^2 + \omega_0^2}$$

whereas if $\omega = \omega_0$ *it is*

$$\frac{t\sin\omega t}{2\omega} \,.$$

The functions $\cos\omega t$, $\cos\omega_0 t$ are both periodic, although with different periods. The function $t\sin\omega t$ oscillates with increasing amplitude as $t \to \infty$. It is a little difficult to imagine exactly how the two kinds of solutions relate to each other, and for that reason it is instructive to graph some examples.

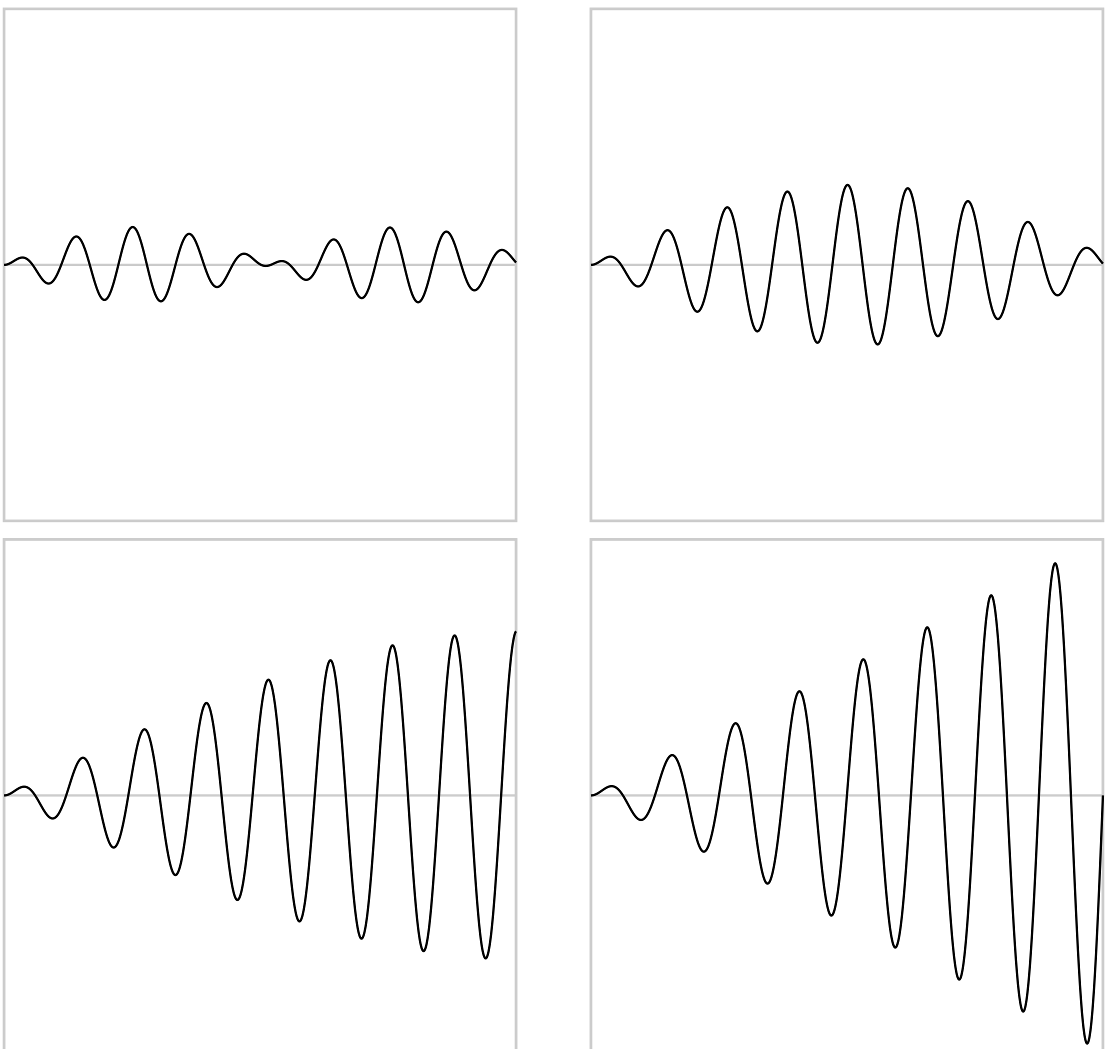


Figure 3.1. *The approach to resonance:* $\omega_0 = 1$ *and* $\omega = 1.24, 1.12, 1.06, 1$.

Why do these pictures look like they do? Recall from trigonometry that

$$\begin{aligned}\cos(x+y) &= \cos x\cos y - \sin x\sin y \\ \cos(x-y) &= \cos x\cos y + \sin x\sin y \,.\end{aligned}$$

We write

$$\omega t = \frac{\omega t + \omega_0 t}{2} + \frac{\omega t - \omega_0 t}{2}$$
$$\omega_0 t = \frac{\omega t + \omega_0 t}{2} - \frac{\omega t - \omega_0 t}{2}$$

which lead to

$$\cos \omega t = \cos \left(\frac{\omega + \omega_0}{2}\right) t \; \cos \left(\frac{\omega - \omega_0}{2}\right) t - \sin \left(\frac{\omega + \omega_0}{2}\right) t \; \sin \left(\frac{\omega - \omega_0}{2}\right) t$$
$$\cos \omega_0 t = \cos \left(\frac{\omega + \omega_0}{2}\right) t \; \cos \left(\frac{\omega - \omega_0}{2}\right) t + \sin \left(\frac{\omega + \omega_0}{2}\right) t \; \sin \left(\frac{\omega - \omega_0}{2}\right) t$$
$$\cos \omega t - \cos \omega_0 t = -2 \sin \left(\frac{\omega + \omega_0}{2}\right) t \; \sin \left(\frac{\omega - \omega_0}{2}\right) t \; .$$

If ω is near ω_0 we can write $\omega - \omega_0 = 2\varepsilon$ where ε is small. Then

$$\left(\frac{\omega + \omega_0}{2}\right) = \omega_0 + \varepsilon$$

and

$$\frac{\cos \omega t - \cos \omega_0 t}{-\omega^2 + \omega_0^2} = \frac{2 \sin \varepsilon t \; \sin(\omega_0 + \varepsilon) t}{(\omega - \omega_0)(\omega + \omega_0)}$$

which is approximately

$$\frac{2 \sin \omega_0 t \; \sin \varepsilon t}{(2\varepsilon)(2\omega_0)} = \frac{\sin \varepsilon t}{\varepsilon} \; \frac{\sin \omega_0 t}{2\omega_0} = \frac{t \sin \varepsilon t}{\varepsilon t} \; \frac{\sin \omega_0 t}{2\omega_0}$$

since ε is small. The first term oscillates rather slowly while the second oscillates very rapidly, so this explains the graphs we see in the first parts of the figure, which portray a rapid oscillation whose amplitude oscillates slowly. Furthermore, as ε goes to 0 the second term has t as limit (for any fixed value of t), which explains why the successive figures look more and more like the last one.

In real life true resonance can never occur—there will always be some friction in the system and we can never match frequencies exactly. But it is a good ideal model for what we do see often in practice.

Exercise 3.1. *Plot*

$$\cos 1.05\, t - \cos t$$
$$2 \cos 1.05\, t - \cos t$$
$$3 \cos 1.05\, t - \cos t$$

Exercise 3.2. *Find approximate values for the relative maximum and minimum of the* **envelope** *curves of the graph*

$$y = A \cos \omega_1 t - B \cos \omega_2 t$$

if ω_1 and ω_2 are close.

4. Energy

The reason resonance occurs is because the energy being fed into a system is synchronized with the system's motion. Think of a swinging pendulum—if you hit it at the wrong frequency, the system will transfer energy back to you (!) while if you hit it right it will retain the energy you feed into it, except for a steady friction loss, and you will be able to increase the amplitude of its swing.

We can see how this works by looking at the differential equation If we multiply it by y' we get

$$my'y'' + kyy' = F(t)y' - c(y')^2 \; .$$

The left hand side is

$$\frac{d}{dt}\left[\frac{mv^2}{2}+\frac{ky^2}{2}\right]=\frac{dE}{dt}$$

since the total energy of the system is the sum of kinetic and potential energies

$$E=\frac{mv^2}{2}+\frac{ky^2}{2} \qquad (v=y')\ .$$

On the right hand side is the rate of work done by the input force $F(t)$ since

$$F(t)dy=F(t)\,\frac{dy}{dt}\,dt$$

is the work done (force times distance) in time dt. There is also a term which can be interpreted as the rate of energy loss due to friction.

Exercise 4.1. *For the free system $my''+cy'+ky=0$, what is the rate of energy loss?*

Exercise 4.2. *For the system $my''+cy'+ky=0$, if it starts from rest at $y=y_0$, how long does it take for the energy to drop by half?*