Chapter 13. Linear systems

In this chapter we shall look in detail at the relationship between eigenvalues and eigenvectors of matrices, and the solutions of homogeneous linear systems with constant coefficients. We shall also look at inhomogeneous systems.

1. Solving systems of differential equations

We begin with a review.

A **system of first order differential equations** is a set of differential equations involving a number of functions:

$$\begin{aligned} y_1' &= f_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= f_2(t, y_1, y_2, \ldots, y_n) \\ &\ldots \\ y_n' &= f_n(t, y_1, y_2, \ldots, y_n) \ . \end{aligned}$$

The number of equations will always be the same as the number of unknown functions.

We shall say something later about arbitrary systems of such equations, but for the moment we shall look only at the simplest kinds, the **linear systems**. In these, the right hand side are linear functions of the y_i, and we shall in fact assume that the coefficients of the functions $y_i(t)$ are constants. The system looks like this:

$$\begin{aligned} y_1' &= a_{1,1}y_1 + a_{1,2}y_2 + \cdots + a_{1,n}y_n + c_1(t) \\ y_2' &= a_{2,1}y_1 + a_{2,2}y_2 + \cdots + a_{2,n}y_n + c_2(t) \\ &\ldots \\ y_n' &= a_{n,1}y_1 + a_{n,2}y_2 + \cdots + a_{n,n}y_n + c_n(t) \ . \end{aligned}$$

If we make up vectors whose coefficients are functions of t

$$y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \ldots \\ y_n(t) \end{bmatrix}, \qquad c = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \ldots \\ c_n(t) \end{bmatrix}$$

then we can write the system as

$$y' = Ay + c$$

where

$$A = [\, a_{i,j} \,] \ .$$

We shall first look at the homogeneous case when $c = 0$.

2. Homogeneous systems of linear differential equations with constant coefficients

We have seen already the basic principle of solving systems of linear differential equations with constant coefficients, suggested by linearity together with time invariance. To see this from another perspective, we look at the simplest case

$$y' = Ay$$

where A is diagonal:

$$\begin{aligned} A &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \\ y &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \ . \end{aligned}$$

The system of differential equations amounts to two equations which are in fact totally independent of each other, and are said to be **decoupled**:

$$y_1' = a_1 y_1$$
$$y_2' = a_2 y_2 \ .$$

We can solve each one separately, without worrying about the other:

$$y_1 = c_1 e^{a_1 t}$$
$$y_2 = c_2 e^{a_2 t}$$

where the coefficients c_1 and c_2 are arbitrary. In vector notation we can write this

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{a_1 t} \\ c_2 e^{a_2 t} \end{bmatrix} = \begin{bmatrix} c_1 e^{a_1 t} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 e^{a_2 t} \end{bmatrix} = c_1 e^{a_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{a_2 t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ .$$

It turns out that any system of first order differential equations nearly always has a solution looking something like this. To be precise:

- *The solutions of this system satisfy the* **linearity principle**—*linear combinations of solutions are again solutions.*
- *If*

  $$y' = Ay$$

  is any system of linear equations with A a matrix whose coefficients are constants, then (a) there exists at least one solution of the form

  $$y = e^{\lambda t}\xi$$

  and (b) we can nearly always find numbers λ and vectors ξ such that the solutions to the system are made up of linear combinations of vector-valued functions of this form.

Finding the possible λ and ξ is simple in principle but often difficult in practice. Suppose that $y = e^{\lambda t}\xi$ is a solution of the system. Then on the one hand because ξ has constant coefficients

$$y' = \lambda e^{\lambda t}\xi = \lambda y$$

and on the other

$$y' = Ay$$

and these two expressions for A are compatible precisely when

$$A\xi = \lambda\xi$$

or, in other words, ξ is an **eigenvector** for A and λ is its **eigenvalue**.

Another way to look at this is that we want to choose different coordinates in order to decouple the equations. The eigenvectors becomes axis vectors in the new system.

Continue with the matrices we looked at earlier.

Example. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \ .$$

The solutions to the system of differential equations

$$y' = Ay$$

are in this case

$$c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Example. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

The solutions to the differential equation are therefore

$$c_1 e^{(1+i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} .$$

We can find solutions with only real coordinates by taking suitable special combinations to obtain the real and imaginary parts of the basic solution. If we take $c_1 = 1/2$, $c_2 = 1/2$ then we obtain the real part, and if we take $c_1 = 1/2i$, $c_2 = -1/2i$ we obtain its imaginary part. Therefore we also have solutions

$$e^{(1+i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^t(\cos t + i \sin t) \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^t \cos t + ie^t \sin t \\ e^t \sin t - ie^t \cos t \end{bmatrix}$$

which are

$$e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad e^t \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

Example. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^2 - 2\lambda + 1 = 0 .$$

Here there is only one eigenvalue $\lambda = 1$ with eigenvector

$$\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

We get from this a solution

$$e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

How do we get another?

This case is somewhat analogous to what happens for second order equations when the characteristic polynomial has two roots. It urns out that there exists a solution of the form

$$y = te^t \xi + e^t \eta .$$

Then

$$y' = e^t \xi + te^t \xi + e^t \eta$$

and if this is to be equal to Ay we must have

$$A(t\xi + \eta) = \xi + t\xi + \eta$$

and if we compare terms we must have

$$A\xi = \xi, \quad A\eta = \xi + \eta \ .$$

Given A, we can go the other way. First we choose ξ to be an eigenvector, and then we solve

$$(A - I)\eta = \xi$$

to find η. If

$$\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then we have to solve the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which leads to the condition $y = 1$. The value of x is arbitrary. We can set it equal to 0. This gives

$$\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so the extra solution is

$$te^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} te^t \\ e^t \end{bmatrix}$$

Let's summarize what can happen for a 2×2 system.

- *When the eigenvalues of A are real and distinct, the solutions of the system are the linear combinations*

$$c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2 \ .$$

- *When A has a complex eigenvalue λ and complex eigenvector ξ we get as real solutions the real and imaginary parts of*

$$e^{\lambda t}\xi$$

where we calculate $e^{\lambda t}$ using Euler's equation

$$e^{a+ib} = e^a(\cos b + i \sin b) \ .$$

- *When A has a single eigenvalue and only a single line of eigenvectors we get one solution*

$$e^{\lambda t}\xi$$

where ξ is an eigenvector and another solution

$$te^{\lambda t}\xi + e^{\lambda t}\eta$$

where

$$(A - \lambda I)\eta = \xi \ .$$

Exercise 2.1. *Find the general solution of the systems*

$$y' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} y$$

Exercise 2.2. *A certain electric circuit is described by the equations*

$$\frac{d}{dt} \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}$$

Suppose $R = 1$ ohm, $C = 1/2$ farad, $L = 1$ henry. Find the general solution of the system.

3. Reducing any system to one of first order

We have looked so far only at systems of first order equations in detail, but these are not always what the laws of physics give us directly. This is not a serious problem, since in fact

- *Any system of differential equations can be transformed into a possibly larger one of first order.*

Rather than explain completely, I will explain how to do this in the simplest case.

Suppose we are given a linear second order differential equation

$$y'' + ay' + by = c(t) \ .$$

Physically, y will often represent position. Let v be y', which will then represent velocity. The equation above can be rewritten in two parts

$$\begin{aligned} y' &= v \\ v' &= y'' \\ &= -av - by + c(t) \ . \end{aligned}$$

But the first and last equations make up a system of two equations of first order.

- *A single equation of second order can be transformed to two of first order.*
- *A system of two equations of second order can be transformed to a system of four equations of first order.*
- *A single equation of any order n can be transformed into n equations of first order.*

This means that if we understand how to solve systems of first order equations then we know how to deal with all systems of any order. There are other ways to use this trick, for example in generalizing our numerical methods of solution to higher order equations without trouble.

Exercise 3.1. *Write down the first order system corresponding to the pendulum equation*

$$my'' + cy' + g\sin(y/\ell) = F(t) \ .$$

4. Initial conditions in systems

Solving for initial conditions in systems of linear differential equations involves solving systems of algebraic linear equations, just as it did for second order linear equations.

The main theoretical result is that any system of linear differential equations

$$y' = Ay$$

has a unique solution with a specified value at an arbitrary value of t.

Example. We are going to find the unique solution of

$$y' = Ay, \quad y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \ .$$

The general solution, as we have just seen, is in this case

$$c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ .$$

Now we have to find explicit values for c_1 and c_2. We set $t = 0$ and calculate

$$y(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix}$$

so that we have to solve the system of algebraic equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + c_2 &= 0 \end{aligned}$$

which gives

$$\begin{aligned} c_1 &= 1/2 \\ c_2 &= 1/2 \ . \end{aligned}$$

Exercise 4.1. *For each of the following systems, find the solution with initial conditions* $(1, 0)$*. Then* $(0, 1)$*.*

$$y' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} y$$

$$y' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} y$$

5. Inhomogeneous linear systems

There are several methods to solve linear inhomogeneous systems

$$y' = Ay + g(t)$$

where $g(t)$ is a vector of functions

$$g(t) = \begin{bmatrix} g_x(t) \\ g_y(t) \end{bmatrix} .$$

Among them is a formula which generalizes the ones we have seen to solve first and second order linear inhomogeneous systems. The starting point is that we *assume* we know how to solve the corresponding homogeneous system (with $g = 0$). Suppose that $y_1(t)$ and $y_2(t)$ are two independent homogeneous solutions (so these also are vectors with functions as coefficients). Let $\Psi(t)$ be the 2×2 matrix whose columns are the solutions $y_1(t)$, $y_2(t)$. Then the general solution to the inhomogeneous system is

$$c_1 y_1(t) + c_2 y_2(t) + \Psi(t) \int^t \Psi(s)^{-1} g(s)\, ds$$

The first two terms form the general solution to the homogeneous system. It can be written as a matrix product

$$\Psi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

The integrand $\Psi(s)^{-1} g(s)$ is the product of a 2×2 matrix and a column vector, hence another column vector. All these have coefficients which are functions of s, and are usually messy to deal with. At any rate the meaning of the integral is the column vector whose coefficients are the integrals of the coefficients of $\Psi(s)^{-1} g(s)$.

This formula is somewhat similar to the ones we have seen before for first order and second order linear equations. In fact, those earlier formulas are essentially a special case of this one.

It has a version with definite integrals, too. The solution of the inhomogeneous system with initial conditions $(0, 0)$ is

$$\Psi(t) \int_0^t \Psi(s)^{-1} g(s)\, ds$$

Example. If

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

then

$$y_1 = \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

so

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

$$\Psi(s)^{-1} = \frac{e^{4s}}{2} \begin{bmatrix} e^{-s} & -e^{-s} \\ e^{-3s} & e^{-3s} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{3s} & -e^{3s} \\ e^{s} & e^{s} \end{bmatrix} .$$

Let

$$g(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Then

$$\Psi^{-1}(s)g(s) = \frac{1}{2} \begin{bmatrix} e^{3s} & -e^{3s} \\ e^{s} & e^{s} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -e^{3s} \\ 2e^{s} \end{bmatrix}$$

and the integral component in the formula gives

$$\begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \int^t \begin{bmatrix} -e^{3s} \\ 2e^{s} \end{bmatrix} ds = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} -e^{3t}/3 \\ 2e^{t} \end{bmatrix} = \begin{bmatrix} 5/3 \\ 7/3 \end{bmatrix} .$$

Exercise 5.1. *For each of the following A*

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$$

find the general solution of the systems

$$y' = Ay + g(t)$$

where

$$g(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Exercise 5.2. *Where*

$$g(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

6. Inhomogeneous systems and eigenvector decomposition

The formula in the previous section always works, but it is hardly ever the most efficient way to solve inhomogeneous systems. There is one case, in particular, when something much quicker is available. That is when

$$g(t) = e^{ct} g_0$$

where g_0 is a constant vector and c is not an eigenvalue of A. In this case there is a solution

$$y(t) = e^{ct} y_0$$

where y_0 is a constant vector also. If we look for a solution of this form we find

$$\begin{aligned} y'(t) &= ce^{ct} y_0 \\ y' - Ay &= ce^{ct} y_0 - Ae^{ct} y_0 \\ &= -(A - cI)e^{ct} y_0 \end{aligned}$$

and solve

$$\begin{aligned} -(A - cI)e^{ct} y_0 &= e^{ct} g_0 \\ -(A - cI) y_0 &= g_0 \\ y_0 &= -(A - cI)^{-1} g_0 \ . \end{aligned}$$

Note that in order for this formula to work, c cannot be an eigenvalue of A. This method is most often used when the exponential is periodic (and $c = i\omega$).

Example. Try this on the example in the previous section where $c = 0$.

Example. Solve

$$y' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} y + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

We first write the vector $(\cos t, \sin t)$ in a way involving complex exponentials. A general fact will be useful here. It follows from Euler's formula by a simple multiplication .

- $A \cos ct + B \sin ct$ *is the real part of* $(A - iB)e^{ict}$.

Therefore

$$\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

is the real part of

$$e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} \ .$$

The eigenvalues of A are -1 and -3, which are different from i, so for the simple solution we set

$$
\begin{aligned}
y(t) &= \text{ real part of } e^{it} y_0 \\
y_0 &= -(A-i)^{-1} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} -2-i & 1 \\ 1 & -2-i \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ i \end{bmatrix} \\
y_0 &= \frac{1}{(2+i)^2-1} \begin{bmatrix} -2-i & -1 \\ -1 & -2-i \end{bmatrix} \begin{bmatrix} -1 \\ i \end{bmatrix} \\
y_0 &= \frac{1}{2+4i} \begin{bmatrix} 2 \\ 2-2i \end{bmatrix} = \frac{1}{1+2i} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \frac{1-2i}{5} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1-2i \\ -1-3i \end{bmatrix} \\
y(t) &= \text{ real part of } \frac{1}{5} \begin{bmatrix} 1-2i \\ -1-3i \end{bmatrix} e^{it} \\
y(t) &= \text{ real part of } \frac{1}{5} \begin{bmatrix} 1-2i \\ -1-3i \end{bmatrix} (\cos t + i \sin t) \\
&= \frac{1}{5} \begin{bmatrix} \cos t + 2\sin t \\ -\cos t + 3 \sin t \end{bmatrix} \ .
\end{aligned}
$$

To get the general solution, we add the general solution of the homogeneous system to this.

7. Resonance for systems

The general solution of the 2×2 system

$$y' = Ay + e^{i\omega t} g_0$$

is

$$y = c_1 y_1 + c_2 y_2 - e^{i\omega t}(A - i\omega)^{-1} g_0$$

where the first two terms make up the general solution to the homogeneous system

$$y' = Ay \ .$$

In most physical systems, energy considerations require that the solutions to the homogeneous system decay rather than grow with time. This means that the real components of the eigenvalues of A are negative or, just conceivably, 0. If they are negative the solutions to the homogeneous equation all die out exponentially, and the first part of the solution is called its **transient** component, the second the **steady state**.

To understand the steady state solution, we express the vector g_0 in terms of the eigenvectors of A. For simplicity, we will assume that A has two independent eigenvectors, so that we can always do this. Thus we write

$$g_0 = \xi_1 + \xi_2$$

where ξ_i is an eigenvector of A, say with eigenvalue λ_k. Then

$$
\begin{aligned}
(A - i\omega)^{-1} g_0 &= (A - i\omega)^{-1}(\xi_1 + \xi_2) \\
&= (A - i\omega)^{-1}\xi_1 + (A - i\omega)^{-1}\xi_2 \\
&= \frac{\xi_1}{\lambda_1 - i\omega} + \frac{\xi_2}{\lambda_2 - i\omega}
\end{aligned}
$$

The steady state solution is therefore

$$e^{i\omega t}\Big[\frac{-\xi_1}{\lambda_1 - i\omega} + \frac{-\xi_2}{\lambda_2 - i\omega}\Big]$$

It has two components. The magnitude of one of these will be very large if $i\omega$ lies close to one of the eigenvalues λ_k. In the case where energy is conserved in the system, one of the eigenvalues will be $\pm\lambda i$, and when ω is near λ the steady state response amplifies drastically.

The coefficient y_0 of $e^{i\omega t}$ is

$$y_0 = \frac{-\xi_1}{\lambda_1 - i\omega} + \frac{-\xi_2}{\lambda_2 - i\omega}$$

is a two-dimensional complex vector. We know that if $z = x + iy$ is a complex number then

$$|z| = \sqrt{z\overline{z}} = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2+y^2}$$

is a measure of its magnitude. If

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

is a complex vector then

$$\|z\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{z_1\overline{z_1} + z_2\overline{z_2}}$$

is defined to be its **magnitude**. In understanding the response of the system to various kinds of input we are interested in how the magnitude of y_0 changes with ω. Resonance of some kind occurs when this plot has a distinct maximum. If you consider all the possibilities, you can see that a real resonance can occur only when the eigenvalues are conjugate complex numbers and the number $i\omega$ lies close to one of them.

Example. Let's look at a new electric circuit:

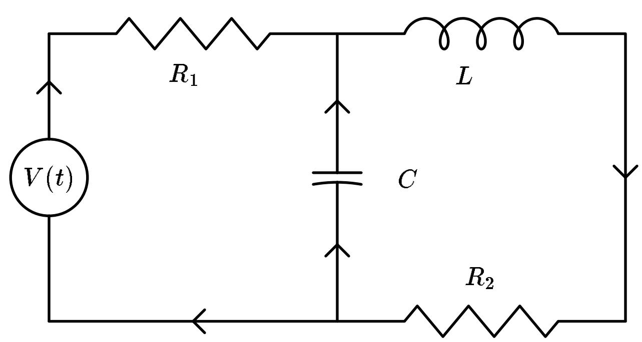

Figure 7.1. *An electric circuit with voltage source.*

I remind you of the defining properties of the various components.

Type of element	Defining property
Resistor	$V = RI$ (Ohm's Law)
Inductor	$V = LI'$
Capacitor	$Q = CV$
	$I = CV'$
Independent voltage source	$V = V(t)$ is specified as a function of time

Let V be the voltage drop across the capacitor, I the current through the coil.

Kirchhoff's Laws tell us

$$\begin{aligned} -I_1 + I - I_C &= 0 \\ I_1 &= I - I_C \\ &= I - CV' \\ V &= -LI' - R_2 I \\ &= V(t) + R_1 I_1 \\ &= V(t) + R_1(I - CV') \\ &= V(t) + R_1 I - R_1 CV' \end{aligned}$$

from which we get the inhomogeneous system

$$\begin{aligned} I' &= -\frac{1}{L} V - \frac{R_2}{L} I \\ V' &= -\frac{1}{R_1 C} V + \frac{1}{C} I + \frac{1}{R_1 C} V(t) \end{aligned}$$

or in matrix form

$$\begin{bmatrix} V \\ I \end{bmatrix}' = \begin{bmatrix} -\dfrac{1}{R_1 C} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R_2}{C} \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_1 C} V(t) \\ 0 \end{bmatrix} .$$

Set $L = C = R_1 = R_2 = 1, V(t) = \cos \omega t$. The system becomes

$$\begin{bmatrix} V \\ I \end{bmatrix}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} + \begin{bmatrix} \cos \omega t \\ 0 \end{bmatrix}$$

and the characteristic polynomial is

$$\lambda^2 + 2\lambda + 2 = 0$$

with roots $-1 \pm i$. The corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

and we can write

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} 1 \\ i \end{bmatrix} + \begin{bmatrix} 1 \\ -i \end{bmatrix} \right) .$$

Since the eigenvalues have negative real part, the solutions to the homogeneous are transient, and the steady state solution is the real part of

$$-\frac{e^{i\omega t}}{2} \left(\frac{1}{-1 + i - i\omega} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{-1 - i - i\omega} \begin{bmatrix} 1 \\ -i \end{bmatrix} \right) .$$

The coefficient y_0 of $e^{i\omega t}$ is

$$y_0 = -\frac{1}{2} \left(\frac{1}{-1 + i - i\omega} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{-1 - i - i\omega} \begin{bmatrix} 1 \\ -i \end{bmatrix} \right)$$

which is a two-dimensional complex vector. We are interested in its magnitude. The coordinates of y_0 can be calculated explicitly but the expression you get is rather messy and not illuminating. It is easier to write a program to calculate its magnitude as a function of ω over a reasonable range of values of ω, than it is to do by hand.

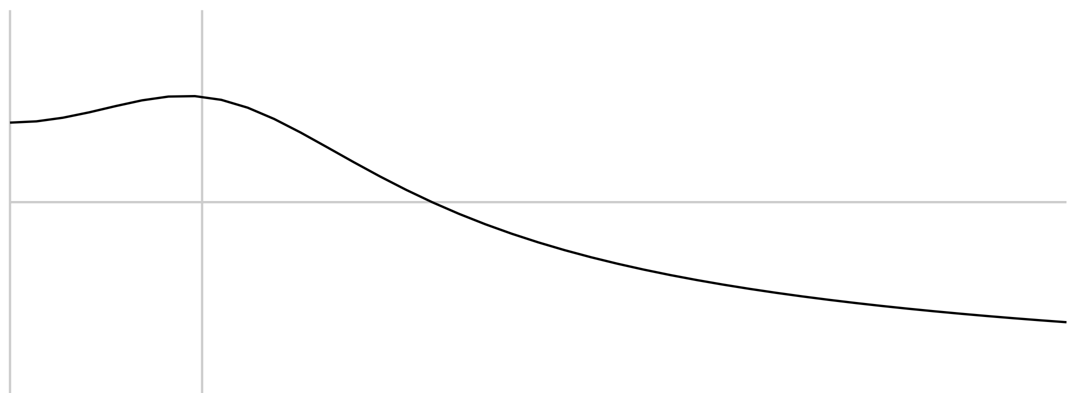

Figure 7.2. *Resonance in a system with two degrees of freedom.*

The resonance is rather weak.

A system of size $n \times n$ can have up to $n/2$ resonances.

8. Eigenvalues and eigenvectors for larger matrices

The main difference between the 2×2 case and that of larger systems is the amount of work involved in finding eigenvalues and eigenvectors. There are also many different cases analogous to the single unusual case for 2×2 matrices when there do not exist linearly independent eigenvectors. The most important case, however, is when the matrix is symmetric, in which case there are relatively few problems beyond the difficulty of finding eigenvalues and eigenvalues. This is a job for computers.

9. Why linear systems are ubiquitous

Why are linear systems so common? It is for a purely mathematical reason. We have seen that for two weights on a spring, the potential energy of the system is that stored in the displacement of the weights away from equilibrium. This energy is stored in the stretched springs, and (up to a constant related to energy in the equilibrium position itself) is given by the expression

$$(1/2)[k_0 y_0^2 + k_1 (y_1 - y_0)^2]$$

where y_i is displacement away from equilibrium. This can be rewritten as

$$(1/2)[(k_0 + k_1) y_0^2 - 2k_1 y_0 y_1 + k_1 y_1^2] \ .$$

It is a **quadratic** expression in y_0 and y_1. Now for any physical system in equilibrium whatsoever, potential energy is at a minimum. This means, by calculus, that the first derivatives of potential energy with respect to the displacement variables vanish. This in turn means that the Taylor series for potential energy must start off

$$P_0 + \text{quadratic terms} + \text{higher order terms}$$

and is therefore well approximated by the expression

$$P_0 + \text{quadratic terms}$$

near equilibrium. If we assume that friction is negligible, then for a physical system of this kind we have Lagrange's form of Newton's law in terms of energy—the system of differential equations

$$\frac{d}{dt} \frac{\partial\, \mathrm{KE}}{\partial v_i} = -\frac{\partial\, \mathrm{PE}}{\partial x_i} \ .$$

The left hand side is often a purely quadratic expression in velocities. If the right hand side is a quadratic expression in position variables, then the system of differential equations we get will be linear. In other words

- *It is because potential energy is at a minimum for physical systems in equilibrium that the behaviour of those systems near equilibrium is approximated by sets of linear differential equations.*