Mathematics 257—January 16, 1996

Second homework solutions

Read:

§10.2	Exercise #29
§§10.3–4	(again, perhaps)

Exercises:

§10.3	6, 9, 10, 11, 15
§10.4	13, 15, 17, 21, 22, 25, 37

Also:

Exercise 1. *(a) Use the identities*

$$\sin(at + bt) = \sin at \cos bt + \cos at \sin bt$$
$$\sin(at - bt) = \sin at \cos bt - \cos at \sin bt$$

to find a formula for

$$\int \cos at \sin bt \, dt$$

(b) Show by calculation that

$$\int_0^T \cos 2\pi m \, \frac{t}{T} \sin 2\pi n \, \frac{t}{T} \, dt = 0$$

for all positive integers m, n.

(a) If we subtract the two formulas we get

$$\cos at \sin bt = \frac{1}{2}(\sin(a + b)t - \sin(a - b)t) \ .$$

If $a \neq b$ this leads to

$$\int \cos at \sin bt \, dt = \frac{1}{2} \left(- \frac{\cos(a + b)t}{(a + b)} + \frac{\cos(a - b)t}{(a - b)} \right)$$

and if $a = b$

$$\int \cos at \sin at \, dt = - \frac{\cos 2at}{4a}$$

If $a = 2\pi m/T$ and $b = 2\pi n/T$ then all terms are periodic of period T, and therefore they cancel at the end points 0 and T.

Exercise 2. *Suppose that f and g are functions of period T. Recall that the dot product of f and g is defined to be*

$$f \bullet g = \int_0^T f(t)g(t) \, dt \ .$$

Let A be the linear operator taking f to f''.

(a) Use integration by parts to show that

$$Af \bullet g = f \bullet Ag$$

(b) Show that each of the functions 1, $\cos 2\pi m(t/T)$, $\sin 2\pi m(t/T)$ (where m is a positive integer) is an eigenvector for A. What are the eigenvalues?

(b) The double derivative of a constant vanishes, so it has eigenvalue 0. If $f(t) = \cos(2\pi mt/T)$ then

$$f''(t) = -\left(\frac{2\pi m}{T}\right)^2 \cos 2\pi m\frac{t}{T} = -\left(\frac{2\pi m}{T}\right)^2 f(t)$$

so it has eigenvalue

$$-\left(\frac{2\pi m}{T}\right)^2$$

It is the same for $\sin(2\pi mt/T)$.

(a) We have

$$Af \bullet g = \int_0^T f''(t)g(t)\,dt$$
$$f \bullet Ag = \int_0^T f(t)g''(t)\,dt$$

and we want to show therefore that

$$\int_0^T f''(t)g(t)\,dt = \int_0^T f(t)g''(t)\,dt$$

But if we apply integration by parts to one of these we get

$$\begin{aligned}\int_0^T f''(t)g(t)\,dt &= \big[f'(t)g(t)\big]_0^T - \int f'(t)g'(t)\,dt \\ &= (f'(T)g(T) - f'(0)g(0)) - \int f'(t)g'(t)\,dt \\ &= -\int f'(t)g'(t)\,dt\end{aligned}$$

since f and g have period T. Since this is an expression which is symmetric in f and g, the other integral is also equal to it.