Mathematics 257—February 5, 1996

Fourth homework solutions

Exercise 1. *Find the solution of the heat equation in 2D*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

on the unit square $0 \le x, y \le 1$ with initial distribution

$$u(0, x, y) = f(x, y) = x$$

and boundary conditions

$$u(t, x, y) = 0 \quad (t > 0)$$

for all (x, y) on the edge of the square.

The eigenvectors for Δ with these boundary conditions are

$$\sin \pi m x \sin \pi n y \quad (m, n > 0)$$

with eigenvalues $c_{m,n} = -\pi^2(m^2 + n^2)$ and if $f(x, y)$ is the initial distribution then the complete solution is

$$u(t, x, y) = \sum_{m,n>0} k_{m,n} e^{\alpha^2 c_{m,n} t} \sin \pi m x \sin \pi n y$$

where

$$k_{m,n} = \frac{\displaystyle\int_0^1 \int_0^1 f(x, y) \sin \pi m x \sin \pi n y \, dx dy}{\displaystyle\int_0^1 \int_0^1 \sin^2 \pi m x \sin^2 \pi n y \, dx dy}$$

The denominator is

$$\int_0^1 \sin^2 \pi m x \, dx \int_0^1 \sin^2 \pi n y \, dy = \frac{1}{4}$$

and in our case $f(x, y) = x$ so

$$k_{m,n} = 4 \int_0^1 x \sin \pi m x \, dx \int_0^1 \sin \pi n y \, dy$$

Since

$$\int x \sin \pi m x \, dx = \frac{-x \cos \pi m x}{\pi m} + \frac{\sin \pi m x}{\pi^2 m^2}$$

and

$$\int \sin \pi m y \, dy = \frac{-\cos \pi n y}{\pi n}$$

we have

$$k_{m,n} = 4 \left[\frac{-x \cos \pi m x}{\pi m} + \frac{\sin \pi m x}{\pi^2 m^2} \right]_0^1 \left[\frac{-\cos \pi n y}{\pi n} \right]_0^1 = 4 \left[\frac{(-1)^{m+1}}{\pi m} \right] \left[\frac{(-1)^{n+1} + 1}{\pi n} \right]$$

Exercise 2. *(a) Using separation of variables, describe all the fundamental solutions (eigenvectors) of the same heat equation on the unit square subject to the homogeneous boundary value conditions*

$$u(t,0,y)=0, \quad \frac{\partial u}{\partial y}(t,x,0)=0, \quad \frac{\partial u}{\partial y}(t,x,1)=0, \quad \frac{\partial u}{\partial x}(t,1,y)=0$$

corresponding to a box insulated on three sides and open on the fourth.

(b) Find an estimate for the solution of this system if the initial temperature distribution is a constant 1.

(c) At what time t does the centre of the square reach temperature $1/2$? At that time t, can you estimate (with reasons) where the temperature is highest?

The fundamental solutions are

$$X(x)Y(y)$$

where

$$X''=cX, \quad X(0)=0=X'(1)$$

and

$$Y''=dY, \quad Y'(0)=Y'(1)=0$$

The solutions are

$$X(x)=\sin\alpha x$$
$$Y(y)=\cos\beta y$$

where since $X'(1)=0$, $Y'(1)=0$

$$\cos\alpha=0, \quad \sin\beta=0$$

which means that

$$\alpha=\pi/2+\pi m, \quad \beta=\pi n \qquad (m\geq 0, n\geq 0)$$

Therefore we have the fundamental solutions

$$\sin(\pi/2+\pi m)x\cos\pi ny \quad (m\geq 0, n\geq 0)$$

with

$$c_{m,n}=-\pi^2(1/2+m)^2+\pi^2n^2=\frac{\pi^2}{4}\left((2m+1)^2+(2n)^2\right)$$

(b) The solution is

$$\sum_{m,n}k_{m,n}e^{\alpha^2 c_{m,n}t}\sin(\pi/2+\pi m)x\cos\pi ny$$

where

$$k_{m,n}=\frac{\displaystyle\int_0^1\int_0^1\sin(\pi/2+\pi m)x\cos\pi ny\,dxdy}{\displaystyle\int_0^1\int_0^1\sin^2(\pi/2+\pi m)x\cos^2\pi ny\,dxdy}$$
$$=4\int_0^1\sin(\pi/2+\pi m)x\,dx\int_0^1\cos\pi ny\,dy$$

which is 0 unless $n=0$ and then equals

$$4\left[\frac{-\cos(\pi/2+\pi m)x}{(\pi/2+\pi m)}\right]_0^1=\frac{4}{\pi/2+\pi m}$$
$$=\frac{8}{\pi(2m+1)}$$

since $\cos(\pi/2 + \pi m) = 0$ for all m. So

$$u(t, x, y) = \sum_{m \geq 0} \frac{8}{\pi(2m+1)} e^{-\alpha^2 \pi^2 (2m+1)^2 t/4} \sin(\pi/2 + \pi m)x$$

and in particular doesn't vary with y.

(c) We approximate by one term, the first, which is

$$\frac{8}{\pi} e^{-\alpha^2 \pi^2 t/4} \sin(\pi/4)$$

so solve

$$\frac{8}{\pi} e^{-\alpha^2 \pi^2 t/4} \frac{\sqrt{2}}{2} = \frac{1}{2}$$

for t. This gives

$$t = \frac{-4 \log(\pi \sqrt{2}/16)}{\alpha^2 \pi^2}$$