Mathematics 257—February 8, 1996

Fifth homework solutions

Determine as well as you can all eigenvalues and eigenfunctions for these problems (note that I have changed the sign of a few):

11.2.2.

$$y'' = \lambda y, \quad y'(0) = 0, \quad y(1) = 0$$

11.2.4.

$$y'' = \lambda y, \quad y(0) = 0, \quad y'(1) = 0$$

11.2.5.

$$y'' = \lambda y, \quad y(0) + y'(0) = 0, \quad y(1) = 0$$

11.2.6.

$$y'' = \lambda y, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0$$

It can be shown directly as in the notes that all eigenvalues are non-positive. If $\lambda = 0$ is an eigenvalue then the eigenfunction must be linear. This is possible only in #5, where $(1 - x)$ is an eigenfunction.

So now we must look for the negative eigenvalues. In these cases $\lambda = -\omega^2$, and the possible eigenfunctions are of the form

$$y(x) = A\cos\omega x + B\sin\omega x$$
$$y'(x) = -A\omega\sin\omega x + B\omega\cos\omega x$$

#2.

$$y'(0) = B\omega$$
$$y(1) = A\cos\omega + B\sin\omega$$

so that $B = 0$, y is a multiple of $\cos\omega x$, and $\cos\omega = 0$.

#4.

$$y(0) = A\omega$$
$$y'(1) = -A\sin\omega + B\omega\cos\omega$$

so that $A = 0$, y is a multiple of $\sin\omega x$, and $\cos\omega = 0$.

#5.

$$y(0) + y'(0) = A + B\omega$$
$$y(1) = A\cos\omega + B\sin\omega$$

so that

$$A + B\omega = 0$$
$$A\cos\omega + B\sin\omega = 0$$

and consequently

$$\det\begin{bmatrix} 1 & \omega \\ \cos\omega & \sin\omega \end{bmatrix} = \sin\omega - \omega\cos\omega = 0$$

or

$$\omega = \frac{\sin\omega}{\cos\omega} = \tan\omega$$

#6. Here $y(0) = A$ so $y = \sin \omega x$ and

$$y(\pi) + y'(\pi) = \sin \pi\omega + \omega \cos \pi\omega$$

so that

$$\sin \pi\omega + \omega \cos \pi\omega = 0$$

or

$$\frac{\sin \pi\omega}{\cos \pi\omega} = -\omega$$

11.2.9. *Consider*

$$y'' - 2y' + (1 + \lambda)y = 0, \quad y(0) = 0, \quad y(1) = 0$$

(a) *Set $y = s(x)u$, a new dependent variable. Determine $s(x)$ so the new equation has no u' term.*

If $y = su$ then

$$y' = s'u + su'$$
$$y'' = s''u + 2s'u' + su''$$

so the differential equation becomes

$$s''u + 2s'u' + su'' - 2s'u - 2su' + (1 + \lambda)su = u''s + u'(2s' - 2s) + u(\lambda s + s - 2s' + s'')$$

so if $s' = s$ or $s = e^x$ the central term vanishes. The equation then becomes (!)

$$e^x(u'' + \lambda u) = 0$$

(b) *Find eigenvalues and eigenfunctions.*

Easy.

11.2.13. *Consider the problem*

$$y'' + \lambda y = 0, \quad 2y(0) + y'(0) = 0, \quad y(1) = 0$$

(a) *Find determinantal equation satisfied by positive eigenvalues. Show that there are an infinite number of them and that*

$$\lambda_n \sim \frac{(2n + 1)\pi}{2}$$

for large n.

Straightforward.

(b) *Find the equation for negative eigenvalues. Show there is exactly one.*

Set $\lambda = -\omega^2$ so we are looking for solutions

$$y(x) = Ae^{\omega x} + Be^{-\omega x}$$
$$y'(x) = A\omega e^{\omega x} - B\omega e^{-\omega x}$$

We get equations

$$2A + 2B + A\omega - B\omega = 0$$
$$Ae^{\omega} + Be^{-\omega} = 0$$

and consequently

$$\det \begin{bmatrix} 2 + \omega & 2 - \omega \\ e^{\omega} & e^{-\omega} \end{bmatrix} = (2 + \omega)e^{-\omega} - (2 - \omega)e^{\omega} = 0$$

or

$$\frac{\omega}{2} = \frac{e^{\omega} - e^{-\omega}}{e^{\omega} + e^{-\omega}}$$

which has one solution $\omega = 1.91500805$.