Mathematics 257—January 24, 1996

The heat equation in higher dimensional regions

The heat equation in any dimension is written

$$\frac{1}{\alpha^2}\frac{\partial u}{\partial t} = \Delta u$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2}$$

if the number of spatial dimensions is one,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

if the number of space dimensions is two, and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

if it is three.

The sort of heat problems we shall discuss here are concerned with heat flow inside a region R, with extra data. There are three components in all to the problem: (1) the heat equation itself, which controls temperature change inside the region; (2) some boundary conditions, which describe what happens at the boundary of the region; (3) an initial temperature distribution, which specifies the temperature distribution throughout R at time $t = 0$.

The basic approach to this problem is essentially the same as it was in the case of dimension one: (1) we attempt to solve the heat equation by separation of variables, obtaining a set of ordinary differential equations involving some undetermined constants; (2) the possible constants are restricted to a discrete set by imposing the boundary conditions; (3) the complete solution is obtained as an infinite sum, something like Fourier series.

We shall in fact look only at four types of regions: two-dimensional plates which are either square or circular, cubes, and spheres.

I shall begin by looking at one example in detail. Consider a square plate, 1 unit on a side, insulated on its two flat sides, so that the heat flow is in just two dimensions. We shall assume that for time $t > 0$ the edges of the plate are maintained at $0°$ and that at time $t = 0$ the temperature is a uniform $100°$ throughout.

The first steps

We can proceed for a while without making any assumptions about dimensions, or indeed using any details of the particular problem at hand.

Step (1). We begin with separation of variables. To do this we do not have to make any assumption at first about the number of dimensions involved. We set

$$u = TU$$

in the heat equation, where $T = T(t)$ and U is a function of space variables alone. We get

$$\frac{1}{\alpha^2}T'U = T\Delta U$$

or

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{\Delta U}{U}$$

which leads to the pair of differential equations

$$T' = c\,\alpha^2 T$$
$$\Delta U = c\,U$$

for some undetermined constant c.

The second equation is again a partial differential equation, but I will postpone looking at it.

Step (2). We introduce the boundary conditions. I will assume that these are at least time-invariant, which means that they are satisfied if the function U satisfies them. For example, if we are dealing with boundary conditions asserting that the temperature is fixed at 0° on the boundary, then we get the conditions that $U = 0$ at boundary points.

Let me suppose for simplicity that in fact we are assuming this to be the boundary condition I am interested in.

So by a partial application of separation of variables, we arrive at the point where we are looking for functions U of the space variables alone satisfying

$$\Delta U = c\,U \text{ inside the region}, \quad U \text{ vanishes on the boundary of the region.}$$

Furthermore, of course, we are interested only in the case where U does not vanish identically.

In other words, U is a kind of eigenvector for the operator Δ acting on the space of functions defined in the region concerned, and vanishing identically on its boundary. The constant c is its eigenvalue.

As I have mentioned several times in class, it is this sort of eigenproblem which underlies most of this course, and solving the eigenproblem is the main part of solving not only heat problems but also others we shall see.

Solving the eigenproblem

We have arrived at the point where we are trying to solve

$$\Delta U = c\,U, \quad U \not\equiv 0, \quad U = 0 \text{ on the boundary of the region } R\ .$$

The differential equation is unfortunately still a partial differential equation. We can only try to solve it, too, by separation of variables.

How we do this depends on the nature of the region we are considering. Suppose it is the unit square. Then we set

$$U(x, y) = X(x)Y(y)$$

and obtain

$$X''(x)Y(y) + X(x)Y''(y) = cX(x)Y(y)$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = c$$
$$\frac{X''}{X} = c - \frac{Y''}{Y}$$

so that

$$\frac{X''}{X} = d$$
$$c - \frac{Y''}{Y} = d$$

for some second constant d. We get then the ordinary differential equations and boundary conditions

$$\begin{aligned} X'' &= dX, \quad X(0) = X(1) = 0 \\ Y'' &= (c-d)Y, \quad Y(0) = Y(1) = 0 \ . \end{aligned}$$

Now, however, we have arrived at exactly the same eigenvalue equations we saw for the heat problem in one dimension. The only possibilities are

$$\begin{aligned} X(x) &= \sin \pi m x, \quad d = -\pi^2 m^2 \\ Y(y) &= \sin \pi n y, \quad c - d = -\pi^2 n^2 \\ U(x,y) &= \sin \pi m x \sin \pi n y, \quad c = -\pi^2 (m^2 + n^2) \end{aligned}$$

where m and n are positive integers.

The series development

If we assume that the initial temperature distribution is $f(x,y)$ for (x,y) inside the unit square then we can write

$$f(x,y) = \sum_{m,n} a_{m,n} \sin \pi m x \sin \pi n y$$

where the sum is over all positive integer pairs m, n and the coefficients are calculated by the formula

$$a_{m,n} = \frac{\int_0^1 \int_0^1 f(x,y) \sin \pi m x \sin \pi n y \, dx dy}{\int_0^1 \int_0^1 \sin^2 \pi m x \sin^2 \pi n y \, dx dy} \ .$$

in analogy with the formula for Fourier series. The denominators can be computed as a product of one-dimensional integrals:

$$\int_0^1 \int_0^1 \sin^2 \pi m x \sin^2 \pi n y \, dx dy = \int_0^1 \sin^2 \pi m x \, dx \int_0^1 \sin^2 \pi n y \, dy = \frac{1}{4}$$

In our case we get

$$\begin{aligned} a_{m,n} &= 4 \cdot 100 \int_0^1 \sin \pi m x \, dx \int_0^1 \sin \pi n y \, dy \\ &= 400 \cdot \frac{1 - (-1)^m}{\pi m} \frac{1 - (-1)^n}{\pi n} \\ &= 400 \cdot \frac{4}{\pi^2 m n} \end{aligned}$$

if m and n are both odd, otherwise $a_{m,n} = 0$.