Mathematics 257—January 23, 1996

Heat flow

In these notes I will explain how to set up a mathematical equation governing how a temperature distribution varying in spatial dimensions forces the temperature to vary in time as well. Roughly speaking, (1) a varying spatial temperature distribution causes heat to flow, and (2) when heat flows it causes the temperature at various places to vary in time as well.

There are in fact two corresponding halves to this discussion. In the first, part where we shall see how a varying temperature distribution in an object causes heat to flow, I will work in dimensions one, two or three. In the second part, where we shall see how the flow of heat in turn affects the temperature distribution, I will look only at the somewhat artificial but still interesting case of an essentially one-dimensional object, such as an insulated thin rod.

This discussion overlaps considerably with that in Appendix A of Chapter 10 in Boyce & DiPrima. What distinguishes it from the discussion in the text is that I introduce the heat flux explicitly.

Temperature distribution affects heat flow

Let

$$u(t, x)$$

be the temperature at time t and position x. I want to discuss several possible dimensions at the same time, so x itself may be a set of coordinates in $2D$ or $3D$.

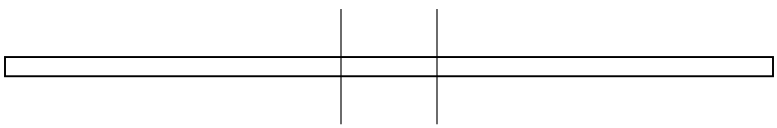
Let κ be the **conductivity** of the substance making up the object. The higher κ is, the more easily heat flows. Silver, for example, is a good heat conductor, and κ will be high.

Very roughly, we can expect that heat flows in the direction in which temperature decreases most rapidly, or opposite to the direction in which it increases most rapidly. The direction in which it increases most rapidly is the direction of the **temperature gradient**. A more precise version of the basic relationship between temperature distribution at any given moment and heat flux is that the heat flux is proportional to the negative temperature gradient. The constant of proportionality turns out to be the conductivity. In a formula

$$H = -\kappa \operatorname{grad}_x u \ .$$

Heat flow causes temperature change

I now assume that we are working in only one spatial dimension.



It is quite possible to say what happens in $2D$ or $3D$, but more complicated, and most of the interesting phenomena occur already in the simpler case. The basic idea here is that as heat flows around inside the object, in time the temperature will rise in some places and decrease in others. Whether it rises or falls will depend whether more heat flows towards a point or away from it. The physical constant here is the **specific heat** s, which determines how much heat it takes to raise a certain amount of the substance by one degree. The way it is used is to say that it takes s units of heat to make a unit mass of the substance to rise one

degree. Water has a high specific heat, silver a low one. The specific heat is a quantity independent of the conductivity—it has nothing to do with how rapidly heat flows in the substance, but only with the effect of a given quantity of heat on temperature, independently of how long it might take to have that effect. The way in which heat flow changes temperature is a bit more subtle than it might first appear—it is not heat flow which changes temperature, but *differential flow* which does that. What I mean by this is that if the heat flow at one end of an interval is the same as the heat flow at the other, then there will be no overall temperature change inside the interval since there is no heat added or subtracted to it. The following equation makes this relationship precise:

$$\frac{\partial u}{\partial t} = \frac{1}{\rho s}\,\frac{\partial H_x}{\partial x}$$

Note that in $1D$ the vector field H has just one component. In higher dimensions, the corresponding equation is also rather simple. In $2D$ it is

$$\frac{\partial u}{\partial t} = \frac{1}{\rho s}\,\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}\right)$$

and in $3D$

$$\frac{\partial u}{\partial t} = \frac{1}{\rho s}\,\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + + \frac{\partial H_z}{\partial z}\right)$$

The extra terms in the right hand side are there simply because there are more dimensions in which heat can flow in and out of any region.

The heat equation

The distribution of temperature at any given moment determines the heat flux at that moment, according to the first equation. The heat flow causes the temperature to vary in time. If we combine the two equations, assuming that conductivity, density, and specific heat are constant through the object, then we get

$$\frac{\partial u}{\partial t} = \alpha^2\,\frac{\partial^2 u}{\partial x^2}$$

where I follow the text in writing

$$\alpha^2 = \kappa/\rho s$$

This single partial differential equation is called the **heat equation**. In any number of dimensions it is

$$\frac{\partial u}{\partial t} = \alpha^2\,\Delta u$$

where Δ is the **Laplacian**

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

In looking at heat flow in $1D$, we shall always be looking at a specific problem of the following sort: we are given (1) a one dimensional object of length ℓ, (2) an initial temperature distribution inside the object, and (3) certain *boundary conditions* describing the physics of heat flow at the end points of the object. We are then required to tell what the temperature distribution is inside the object at all times $t > 0$. In looking at heat flow problems in higher dimensions, only the nature of the boundary assumptions becomes significantly more complicated, basically since the boundary of the region involved will be more complicated.