Mathematics 257—February 6, 1996

Midterm examination solutions

(•) *Find the Fourier series of the function $f(t)$ which is t on the interval $[-1,1]$ and repeated periodically outside that interval.*

The function $f(t)$ is odd, so we only have to deal with sin terms. The period is 2, so the functions are $\sin \pi m t$ for $m > 0$.

$$b_m = \frac{\int_{-1}^{1} t \sin \pi m t \, dt}{\int_{-1}^{1} \sin^2 \pi m t \, dt} = 2 \int_0^1 t \sin \pi m t \, dt = \left[\frac{-t \cos \pi m t}{\pi m} + \frac{\sin \pi m t}{\pi^2 m^2} \right]_0^1 = \frac{2(-1)^{(m+1)}}{\pi m}$$

(•) *(a) A bar of unit length made from a material with $\alpha^2 = 1$ is cooled to a uniform temperature of $0°$ and then heated so that both ends are maintained at $100°$ and no heat enters except at the ends. Assuming perfect conductivity at its ends, find the temperature distribution $u(t,x)$ for all $t > 0$ inside the bar.*

(b) Estimate at what time the temperature at the centre of the bar reaches $50°$.

(a) This question can be done in either of two ways. Strictly speaking, it is an inhomogeneous problem, since the steady state temperature is a uniform $u_*(x) = 100°$ throughout the bar. Therefore we now look for a solution to the homogeneous problem with initial temperature $-100°$ and boundary condition that $u(t,x)$ is zero at the ends of the bar. This is a problem we know well. The solution is therefore

$$u(t,x) = 100 + \sum_n k_n \sin \pi n x$$

where

$$\begin{aligned} k_n &= -\frac{100 \int_0^1 \sin \pi n x \, dx}{\int_0^1 \sin^2 \pi n x \, dx} \\ &= -200 \int_0^1 \sin \pi n x \, dx \\ &= 200 \left[\frac{\cos \pi n x}{\pi n} \right]_0^1 \\ &= 200 \left[\frac{(-1)^n - 1}{\pi n} \right] \\ &= \begin{cases} -\frac{400}{\pi n} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) Use a one-term approximation. We get

$$\frac{4}{\pi} e^{-\pi^2 t} = \frac{1}{2}, \quad t = \frac{\log(8/\pi)}{\pi^2} \sim 0.095$$

(•) *Find all functions $f(x,y)$ defined on the unit square $0 \le x, y \le 1$ with*

$$\Delta f = cf$$

if (1) f is not identically 0; (2) $\partial f/\partial x$ vanishes where $x = 0$, $x = 1$; and (3) $\partial f/\partial y$ vanishes where $y = 0$, $y = 1$. What are the possible values of c?

Using separation of variables we express $f(x, y) = X(x)Y(y)$ where

$$\frac{X''}{X} + \frac{Y''}{Y} = c$$

or

$$X'' = dX, \quad Y'' = (c - d)Y$$

and boundary conditions

$$X'(0) = X'(1) = 0, \quad Y'(0) = Y'(1) = 0$$

The solutions are

$$X(x) = \cos \pi m x, \quad Y(y) = \cos \pi n y \quad (m, n \geq 0)$$

and the possible values of c are $-\pi^2(m^2 + n^2)$.