Mathematics 257—February 15, 1996

How to solve partial differential equations

In these notes I will outline the procedure we will use to solve all the **partial differential equations** we will see in Chapter 10. This is the heart of this course, with many physical applications.

I will illustrate the process by looking in detail at one particular example, a heat flow problem in one dimension.

We are going to consider heat flow in a rod of unit length which is insulated on the sides so that all heat flow takes place longitudinally; we are going to require that the temperature is fixed at $0°$ at both ends for times $t > 0$; and we are going to assume that at the beginning (when $t = 0$) the rod has a fixed and uniform temperature distribution $u(0, x) = T$.

Mathematically we have these conditions:

(1) The temperature is a function of one space variable x, and therefore satisfies the **one-dimensional heat equation**

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2} .$$

(2) It satisfies the **boundary conditions**

$$u(t, 0) = u(t, 1) = 0 \quad (t > 0) .$$

(3) It satisfies the **initial condition**

$$u(0, x) = T \quad (0 \le x \le 1) .$$

We shall see that from this data we can find a formula for $u(t, x)$ for all $t > 0$, x in the range $[0, 1]$.

There are three steps in the process of solving this set-up, corresponding to the three components of the data above: (1) we can solve the heat equation by the method called **separation of variables**, *in which we write $u(t, x)$ as a product of functions of t and x alone. This will introduce some mysterious constant. (2) Imposing the boundary conditions will restrict this constant to a certain discrete set of possibilities. (3) Imposing the initial condition will allow us to express the solution as a possibly infinite linear combination of solutions found by combining (1) and (2).*

Step (1). We attempt to solve the partial differential equation by expressing

$$u(t, x) = T(t)X(x)$$

If we substitute this into the heat equation we get

$$T'(t)X(x) = \alpha^2 T(t)X''(x) .$$

If we divide through by $T(t)X(x)$ we can arrange all references to t on one side, all to x on the other (which is why the method is called **separation of variables**). I divide by α^2 as well to make life a bit easier down the road.

$$\frac{1}{\alpha^2} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

This is the crucial point: *the left hand side involces only t, while the right hand side involves only x. This can happen only if both sides are equal to some common constant, say c.* So we have

$$\begin{aligned} T' &= \alpha^2 cT \\ X'' &= cX \end{aligned}$$

So our original partial differential equation has been separated into two ordinary differential equations. We know how to solve these, because they both have constant coefficients. We get (except in unusual circumstances)

$$T = const\, e^{c\alpha^2 t}$$
$$X = const_1\, e^{\lambda t} + const_2\, e^{-\lambda t}$$

where $\lambda = \sqrt{c}$. Note that if c is 0 we are in an unusual circumstance, and that if c is negative we are implicitly involved with sin and cos. I recall that

$$e^{x+iy} = e^x(\cos y + i \sin y) \ .$$

Step (2). The boundary conditions require that

$$X(0) = X(1) = 0 \ .$$

Some easy linear algebra, together with the observations I just made above about the solutions to the equation $X'' = cX$, will lead us to see that the only (non-trivial) possibilities are (up to a scalar multiple)

$$X(x) = X_n(x) = \sin \pi n x$$

for a positive integer n, which means that

$$c = c_n = -\pi^2 n^2$$

so that up to a scalar

$$T(t) = T_n(t) = e^{-\pi^2 n^2 \alpha^2 t} \ .$$

Step (3). In the final step, I claim that the solution $u(t,x)$ we are looking for must be a linear combination of the solutions we now have at hand:

$$\begin{aligned} u(t,x) &= a_1 T_1(t) X_1(x) + a_2 T_2(t) X_2(x) + a_3 T_3(t) X_3(x) + \cdots \\ &= a_1 e^{-\alpha^2\pi^2 t} \sin \pi x + a_2 e^{-4\alpha^2\pi^2 t} \sin 2\pi x + a_3 e^{-9\alpha^2\pi^2 t} \sin 3\pi x + \cdots \end{aligned}$$

for some constants a_1, a_2, etc. How can we calculate these constants? If we set $t = 0$ we get

$$T = a_1 \sin \pi x + a_2 \sin 2\pi x + a_3 \sin 3\pi x + \cdots$$

for x in the range $[0,1]$, and it turns out that this will allow us to calculate the a_i according to the theory of **Fourier series**.

Explicitly we will get

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

so that finally

$$u(t,x) = \frac{2}{\pi}\left(e^{-\alpha^2\pi^2 t} \sin \pi x + \frac{1}{3}\, e^{-9\alpha^2\pi^2 t} \sin 3\pi x + \frac{1}{5}\, e^{-25\alpha^2\pi^2 t} \sin 5\pi x + \cdots \right) \ .$$

To finish up I will fill in the gaps in this argument. They involve simple calculations which you will, however, feel to come out of nowhere. We will see later how they can be motivated.

Finding the possible constants

We want to know what restrictions are placed upon the constant c if $X(x)$ is a non-zero function such that

$$X'' = cX, \quad X(0) = X(1) = 0 \,.$$

The solutions of this ordinary differential equation come in two varieties: (1) $c = 0$. In this case $X = a + bx$ for suitable a, b. But then if we set $a + bx$ equal to 0 at $x = 0$ and $x = 1$ we see that a and b must both be 0. (2) $c \neq 0$. If $\lambda = \sqrt{c}$, then

$$X = ae^{-\lambda x} + be^{\lambda x}$$

for some a and b. Using the boundary conditions we get

$$\begin{aligned} a + b &= 0 \\ ae^{-\lambda} + be^{\lambda} &= 0 \end{aligned}$$

which is a two-by-two system of equations in a and b. Now a two-by-two system like this has a unique solution unless the determinant of the coefficients is equal to 0. The unique solution must be the trivial one with $a = 0$, $b = 0$. So if we have a non-trivial solution we must have the determinant

$$\det \begin{vmatrix} 1 & 1 \\ e^{-\lambda} & e^{\lambda} \end{vmatrix} = e^{\lambda} - e^{-\lambda} = 0$$

so that

$$\begin{aligned} e^{-\lambda} &= e^{\lambda} \\ e^{2\lambda} &= 1 \end{aligned}$$

One possibility is $\lambda = 0$, but this is impossible unless $c = 0$, which we have dealt with before. Otherwise we must allow the possibility that λ be complex, say $\alpha + \beta i$. But then

$$e^{2\lambda} = e^{2\alpha}(\cos 2\beta + i \sin 2\beta)$$

and we must have

$$\alpha = 0, \quad \beta = \pi n$$

for some non-zero integer n, so that $\lambda = \pi n i$, and $c = -\pi^2 n^2$. But then $X(x) = \sin \pi n x$. Note that positive and negative n give essentially the same functions, so we may as well assume $n > 0$.

Finding the coefficients in the infinite sum

We have

$$\begin{aligned} u(t,x) &= a_1 e^{-\alpha^2\pi^2 t} \sin \pi x + a_2 e^{-4\alpha^2\pi^2 t} \sin 2\pi x + a_3 e^{-9\alpha^2\pi^2 t} \sin 3\pi x + \cdots \\ T &= a_1 \sin \pi x + a_2 \sin 2\pi x + a_3 \sin 3\pi x + \cdots \end{aligned}$$

for some constants a_n. How can we calculate them? There is a trick we will shall see many times. To find a_n we multiply both sides of the equation by the function $\sin \pi n x$, and then integrate along the rod:

$$T \sin \pi n x = a_1 \sin \pi x \sin \pi n x + a_2 \sin 2\pi x \sin \pi n x + a_3 \sin 3\pi x \sin \pi n x + \cdots + a_n \sin \pi n x \sin \pi n x + \cdots$$

$$\int_0^1 T \sin \pi n x \, dx = a_1 \int_0^1 \sin \pi x \sin \pi n x \, dx + a_2 \int_0^1 \sin 2\pi x \sin \pi n x \, dx + \cdots + a_n \int_0^1 \sin \pi n x \sin \pi n x \, dx + \cdots$$

We now need this integral formula:

$$\int_0^1 \sin \pi m x \sin \pi n x \, dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

which follows from the trigonometrical formulas

$$\begin{aligned}\cos\pi(m+n)x &= \cos\pi mx\cos\pi nx - \sin\pi mx\sin\pi nx\\ \cos\pi(m-n)x &= \cos\pi mx\cos\pi nx + \sin\pi mx\sin\pi nx\\ \sin\pi mx\sin\pi nx &= \frac{\cos\pi(m-n)x - \cos\pi(m+n)x}{2}\end{aligned}$$

Because of this formula, only a single term on the right hand side is non-zero, and we get

$$T\left[\frac{1-(-1)^n}{\pi n}\right] = a_n/2\ldots$$

and

$$a_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd}\end{cases}$$