Mathematics 257—February 15, 1996

The sign of eigenvalues

We have seen many circumstances where we could rule out positive eigenvalues by explicit calculation. In fact, there is often a general reason why there are none. We will start by looking at one dimensional problems.

Example. Suppose we are looking for solutions of

$$y'' = \lambda y, \quad y(a) = 0, \quad y(b) = 0$$

We know that the possible values of λ are real, because of a symmetricity argument. Let me recall what that involves. We want to know that

$$f'' \bullet g = f \bullet g''$$

if f and g are two functions on the unit interval where $f(a) = f(b) = 0$, and similarly for g. The dot product here is integration over $[a, b]$.

We have

$$\int_a^b f'' g \, dx = [f'g]_a^b - \int_a^b f'g' \, dx$$

We used this before to deduce that

$$f'' \bullet g - f \bullet g'' = f'(b)g(b) - f'(a)g(a) - f(b)g'(b) + f(a)g'(a)$$

so that if f and g vanish at a and b then $f'' \bullet g = f \bullet g''$. Very generally, *we will have symmetricity whenever the boundary conditions imposed guarantee that*

$$f'(b)g(b) - f'(a)g(a) - f(b)g'(b) + f(a)g'(a) = 0$$

But now we make a new application of the integration by parts. If we set $f = g$ in the original equation we have

$$f'' \bullet f = f'(b)f(b) - f'(a)f(a) - \int_a^b \left[f'(x)\right]^2 \, dx$$

so that if f vanishes at a and b then

$$f'' \bullet f - \int_a^b \left[f'(x)\right]^2 \, dx \leq 0$$

If $f'' = \lambda f$ then we must have

$$\lambda \int_a^b f^2(x) \, dx = - \int_a^b \left[f'(x)\right]^2 \, dx$$

Since we are assuming that f itself does not vanish identically, since it is an eigenfunction, then the integral of $f(x)^2$) does not vanish, since it will have at least some area underneath its graph, and we can divide by it to see that $\lambda \leq 0$. Furthermore $\lambda = 0$ only when

$$\int_a^b \left[f'(x)\right]^2 \, dx = 0$$

which implies in turn that $f'(x)$ must vanish everywhere in $[a, b]$. This can happen only if f is a constant, but the constant must be 0 since $f(0) = 0$.

To summarize a part of this argument, if $f'' = \lambda f$ we have

$$\lambda \int_a^b f^2(x)\,dx = f'(b)f(b) - f'(a)f(a) - \int_a^b [f'(x)]\,dx$$

and λ necessarily non-positive whenever the boundary conditions guarantee that

$$f'(b)f(b) - f'(a)f(a) \leq 0$$

Exercise. For which of the following pairs of boundary conditions for the equation

$$f'' = \lambda f$$

does the same argument work to guarantee that $\lambda \leq 0$? Write out the reasons if it does, and if it doesn't find a positive eigenvalue.

(a) $f'(a) = 0, \quad f(b) = 0$

(b) $f'(a) = 0, \quad f'(b) = 0$

(c) $f(a) = 0, \quad f(b) + f'(b) = 0$

(d) $f'(a) = 0, \quad f(b) = 0$

(e) f of period T

Exercise. For which α does this argument show that the problem

$$f'' = \lambda f, \quad \alpha f(0) + f'(0) = 0, \quad f(1) = 0$$

has no positive eigenvalues?

(b) For which α is 0 an eigenvalue?

Exercise. Specify conditions on α and β in order that the problem

$$f'' = \lambda f, \quad \alpha f(0) + f'(0) = 0, \quad \beta f(1) + f'(1) = 0$$

have no positive eigenvalues.

(b) For which α and β is 0 an eigenvalue?

Exercise. Can a problem

$$f'' = \lambda f, \quad \alpha f(0) + f'(0) = 0, \quad f(1) = 0$$

ever have more than one positive eigenvalue?