Mathematics 257—January 25, 1996

Separation of variables and eigenvalues

Essentially every heat problem we shall see in this course has three components:

(1) It concerns heat flow in a homogeneous object, so that temperature distribution u satisfies the **heat equation**

$$\frac{\partial u}{\partial t} = \alpha^2 \Delta u$$

in the region concerned, where Δ is the Laplacian in the appropriate number of variables.

(2) The **initial temperature distribution** is specified to be a given function $f(x)$, etc. in one, two, or three space variables depending on the number of space dimensions involved.

(3) The flow of heat between the region concerned and its environment is described by certain **boundary conditions** which concern only points on the boundary of the region concerned. The simplest of these is that the temperature itself is fixed on the boundary, but there are other possibilities. For example, if the object is insulated entirely from its environment then there will be no heat flow in or out of the boundary, which means that the derivative of the temperature distribution in the direction perpendicular to the boundary will always vanish.

In attempting to solve the problem specified in this way, in every case we set $u = T(t)F$ where T depends only on time t and F is a function of spatial variables alone. We get

$$T'F = \alpha^2 T \, \Delta F$$

or

$$\frac{1}{\alpha^2} \frac{T'}{T} = \frac{\Delta F}{F}$$

and since the left hand side depends only on t while the right hand side depends only on spatial variables, there must exist some constant c such that

$$T' = c\alpha^2 T$$
$$\Delta F = cF$$

The first equation is simple enough to solve:

$$T(t) = Ke^{\alpha^2 ct}$$

for some constant C. It doesn't tell us much about what c can be, except that physical reasoning suggests this observation: temperature must generally decrease in any physical system because its entropy must increase. Therefore we would be surprised unless $c < 0$. But this is not a completely trustworthy argument.

In fact, the equations above cannot by themselves decide anything about c. *We can restrict the possibilities only by taking the boundary conditions into account.* We have seen several examples of this.

Example. Suppose we are looking at a one dimensional rod, say of unit length. Suppose the boundary conditions fix the temperature at the endpoints at 0. If $T(t)F(x)$ is to satisfy these then we must have

$$F(0) = F(1) = 0$$

Theorem. *If*

$$F''(x) = cF(x)$$

and also

$$F(0) = F(1) = 0$$

then $c = -\pi^2 m^2$ *for some positive integer* m *and*

$$F(x) = A \sin \pi m x$$

I have already proven exactly this in class, and I think even in notes, but I'll repeat the argument here. In fact I will sketch two arguments. They are typical of what happens in many examples in this course.

Argument # 1.

To find the general solution of the ordinary differential equation

$$F'' = cF$$

we find the roots of the characteristic polynomial

$$\lambda^2 = c, \quad \lambda = \pm\sqrt{c}\ .$$

Let $\mu = \sqrt{c}$. Set

$$F = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

Here, μ is conceivably complex, since *a priori* we don't know that c isn't complex, and it is certainly possible that $c < 0$ and μ is imaginary. Furthermore, this formula fails in the special case $c = 0$, since here the two roots of the characteristic polynomial coincide, and the general solution is

$$F = c_0 + c_1 x$$

In the special case $c = 0$ it is straightforward to see that any function of the form $c_0 + c_1 x$ which vanishes at 0 and 1 must itself vanish identically. Therefore 0 cannot be an eigenvalue.

In the general case, say $\mu = a + ib$. We expect to to use Euler's formula $e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx$ to evaluate complex exponentials.

$$F(0) = c_1 + c_2$$
$$F(1) = c_1 e^{\mu} + c_2 e^{-\mu}$$

so the boundary conditions lead to a pair of equations

$$c_1 + c_2 = 0$$
$$c_1 e^{\mu} + c_2 e^{-\mu} = 0$$

We know that two linear equations in two unknowns will have a unique solution unless the coefficient matrix is singular. We always have the solution where $c_1 = c_2 = 0$, and we want to know when others exist. This will happen when

$$\begin{vmatrix} 1 & 1 \\ e^{\mu} & e^{-\mu} \end{vmatrix} = e^{-\mu} - e^{\mu} = 0$$

or

$$e^{\mu} = e^{-\mu}$$

or

$$1 = e^{2\mu} == e^{2a}(\cos 2b + i \sin 2b)$$

The number e^{2a} is positive. If z is any complex number

$$z = r(\cos\theta + i\sin\theta)$$

with r positive then (r, θ) are the polar coordinates of z. The polar coordinates of 1 are $(1, 0)$ so we must have

$$e^{2a} = 1, \quad 2b = 2\pi m$$

for some integer m. This means $a = 0$, $b = \pi m$. We know that $m \neq 0$ since if it were $\mu = 0$. Thus $\mu = \pi m$ with $m \neq 0$.

Then finally $c_1 = -c_2$, so if we set $c_1 = c$ we get

$$F(x) = c(e^{\pi mix} - e^{-\pi mix}) = 2ci\sin\pi mx$$

and F must be a multiple of $\sin\pi mx$.

Argument # 2.

This is the same as the first, except that we can avoid complex exponentials. This is because of the formal reasoning about symmetric operators. I have shown in class notes already that the operator $f \mapsto f''$, identifying vectors with those functions on $[0, 1]$ which vanish at the endpoints, is **symmetric**. Therefore we know automatically that its eigenvalues are real, and we need only to look at the three cases

$$c = \omega^2, \quad c = 0, \quad c = -\omega^2$$

instead of arbitrary complex numbers c.

Look at the last case, for example. We are solving

$$F'' = -\omega^2 F$$

which has solutions

$$A\cos\omega x + B\sin\omega x$$

If we put in the boundary conditions we see that $A = 0$ and $\omega = \pi m$ for $m > 0$.

Example. Suppose the operator is $f \mapsto f''$ again, but the boundary conditions are that $f(t + 1) = f(t)$ for all t. Then the eigenvalues are all $-4\pi^2 m^2$ and the eigenvectors are 1, $\cos 2\pi mx$, $\sin 2\pi mx$.

Example. Let the operator be $f \mapsto \Delta f$ for f on the unit sqaure $0 \leq x, y \leq 1$, with boundary conditions that f vanish at the edge of the square. Then we solve the equation

$$\Delta f = cf$$

by separation of variables. Using boundary conditions reduces the problem to the first one we looked at, in both x and y, so the eigenvalues are

$$-\pi^2(m^2 + n^2)$$

eigenvectors

$$\sin\pi mx\sin\pi ny$$

One final remark is that this sort of thing works directly only for **homogeneous** boundary conditions, where something is set to vanish, where *the linear combination of two functions satisfying the boundary condition again satisfies it.*