Mathematics 257—January 12, 1996

Fourier series and symmetric linear operators

Eigenvalues and symmetric matrices

If A is an $n \times n$ matrix, an eigenvector of A is an n-vector $v \neq 0$ such that Av is a scalar multiple of v:

$$Av = cv$$

for some constant c. The constant c is called the eigenvalue of v.

The equation above can be rewritten

$$(A - cI)v = 0$$

where I is the $n \times n$ identity matrix. This gives rise to a system of n linear equations satisfied by the coefficients of v. Now normally n equations in n unknowns have exactly one solution, which here would be the zero vector. In this case, by assumption, the vector v is not the zero vector, which means that the coefficient matrix $A - cI$ must be singular. In other words, the eigenvalues are the roots of the characteristic polynomial of A

$$\det(A - xI) = 0$$

It has degree n, and generally n roots. Given a root of this equation we can find all of the corresponding eigenvectors by finding all solutions of the system

$$(A - cI)v = 0$$

which will make up a linear subspace in n dimensions.

Example. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Then

$$\det(A - xI) = (2 - x)^2 - 1 = x^2 - 4x + 3$$

so the eigenvalues are $c = 1, 3$. This illustrates the first basic afct about the eigenvalues of symmetric matrices:

(1) *The eigenvalues of a symmetric matrix are always real.*

For contrast, consider the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

whose characteristic polynomial is

$$x^2 + 1$$

and whose eigenvalues are $\pm i$.

For $c = 1$ the eigenvector equations of A are

$$(A - I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

leading to the system of equations

$$x + y = 0$$
$$x + y = 0$$

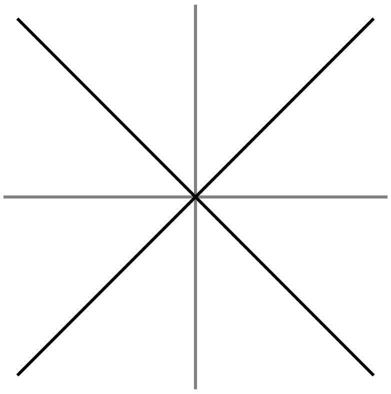
The system is redundant, as it must be, so the eigenvectors for $c = 1$ make up the line

$$x + y = 0$$

For $c = 3$ the eigenvectors make up the line

$$x - y = 0$$

Here is a picture of the eigenvector lines.



The picture illustrates the second basic fact about the eigenvalues of a symmetric matrix:

(2) *If $c_1 \neq c_2$ are eigenvalues of a symmetric matrix with eigenvectors v_1 and v_2, then v_1 and v_2 are perpendicular to each other.*

In higher dimensions, this is equivalent to the condition that the dot product vanish:

$$v_1 \bullet v_2 = 0$$

For the third basic fact, look at the non-symmetric matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Its characteristic polynomial is

$$x^2 - 1$$

and it has only $c = 1$ as an eigenvalue. The eigenvector equations are

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which leads to the single eigenvector equation $y = 0$. In other words, there is only a single line of eigenvectors, namely the x-axis. This does not happen for symmetric matrices:

(3) *If A is an $n \times n$ symmetric matrix, then we can find n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$.*

We may in fact find ones which are all mutually perpendicular, forming an **orthogonal** basis. This means that if v is any n dimensioanl vector then it can be expressed as a linear combination

$$v = x_1 v_1 + x_2 v_2 + \cdots x_n v_n$$

The fourth basic fact we shall need is not really a property of symmetric matrices, but of orthogonal bases in general:

(4) *In the above expression for* v

$$x_m = \frac{v_m \bullet v}{v_m \bullet v_m}$$

This is an easy calculation. Fix m and take the dot product of v_m with both sides of the expression for v. We get

$$\begin{aligned} v_m \bullet v &= v_m \bullet \big(x_1 v_1 + x_2 v_2 + \cdots x_n v_n\big) \\ &= x_1\, v_m \bullet v_1 + x_2\, v_m \bullet v_2 + \cdots x_n\, v_m \bullet v_n \\ &= x_m\, v_m \bullet v_m \\ x_m &= \frac{v_m \bullet v}{v_m \bullet v_m} \end{aligned}$$

since $v_m \bullet v_i = 0$ unless $i = m$.

Proof of orthogonality

Suppose A to be a matrix. It is called **symmetric** when $a_{i,j} = a_{j,i}$ for all $i \neq j$, or equivalently if a flip around the NW-SE diagonal doesn't change it. There is another way to characterize a symmetric matrix in terms of the linear operator it represents, that is to say in terms of the transformation the matrix performs taking v to Av. First note that the *dot* product of two vectors can be written as a *matrix* product

$$x_1 y_1 + x_2 y_2 + \cdots x_n y_n = [\, x_1 \quad x_2 \quad \ldots \quad x_n \,] \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix} = {}^t x\, y$$

if x and y are both taken to be column vectors. (Here ^{t} means transposition.)

A fifth basic fact is that

(5) *The matrix* A *is symmetric precisely when*

$$Ax \bullet y = x \bullet y$$

for all vectors x *and* y.

This is because

$$\begin{aligned} Ax \bullet y &= {}^t(Ax) y \\ &= {}^t x\, {}^t A\, y \\ x \bullet Ay &= {}^t x\, A\, y \end{aligned}$$

and the two expressions are the same only when ${}^t A = A$.

It is this characterization which is responsible for orthogonality and reality of eigenvalues. Suppose v is an eigenvector and c its eigenvalue. At first we have to allow the possibility that c is a complex number and v has complex coefficients. If

$$v = (x_1, x_2, \ldots, x_n)$$

let

$$\overline{v} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$$

be its **conjugate** vector. The conjugate of a complex number

$$x = a + ib$$

is

$$\overline{x} = a - ib$$

Then

$$Av \bullet \overline{v} = cv \bullet \overline{v} = c(v \bullet \overline{v})$$

but because A is symmetric

$$\begin{aligned} & Av \bullet \overline{v} \\ & \quad = v \bullet A\overline{v} \\ & \quad = v \bullet \overline{Av} \\ & \quad = v \bullet \overline{cv} \\ & \quad = \overline{c}\,(v \bullet \overline{v}) \end{aligned}$$

Therefore $\overline{c} = c$.

If c_1, c_2 are distinct eigenvalues for eigenvectors v_1, v_2 then

$$Av_1 \bullet v_2 == c_1(v_1 \bullet v_2) = v_1 \bullet Av_2 = c_2(v_1 \bullet v_2)$$

which implies that $v_1 \bullet v_2 = 0$.

Functions as vectors

The calculation for (4) in the previous section is very similar to the one which found an expression for the Fourier coefficients of a periodic function. This is no accident. A periodic function may be considered as a kind of vector. If f and g are two periodic functions with the same period T then we can define operations analgous to **scalar multiplication**, **vector addition**, and a **dot product**, which are the chacateristic features of vectors.

$$\begin{aligned} [cf](t) &= cf(t) \\ [f+g](t) &= f(t) + g(t) \\ f \bullet g &= \int_0^T f(t)g(t)\,dt \end{aligned}$$

Of these three operations, the dot product seems the most peculiar. But it has many of the properties of the usual dot product. In particular

$$f \bullet f = \int_0^T f(t)^2\,dt$$

which is always non-negative, and is equal to 0 only if $f(t) = 0$ for all t in the interval $[0, T]$, which means that f itself must vanish everywhere because of periodicity.

A linear operator A is called symmetric if

$$Af \bullet g = f \bullet Ag$$

for all periodic functions f, g. For example, the operator taking f to f'' turns out to be symmetric.

The same argument as before shows that the eigenvalues of a symmetric operator are real and that eigenvectors for distinct eigenvalues are orthogonal.