Mathematics 266 — Spring 1999 — Part II

Chapter 1. Introduction to complex-valued functions of a complex variable

The second part of this course is concerned with something that has a few, but not too many, points in common with vector calculus. In electrical engineering this material appears in two ways, one as a supplement to the topic of Laplace transforms, and the other as a tool for analyzing electric fields in 2D.

1. Geometry and complex numbers

The secret to working happily with complex numbers is to think of them as vectors in 2D, so that $z = x + iy$ is a vector with horizontal coordinate x and vertical coordinate y. Associated to it is its length $\|z\| = \sqrt{x^2 + y^2}$ and its polar angle, which is called its **argument**. If w has length r and argument θ then

$$w = r\cos\theta + ir\sin\theta\ .$$

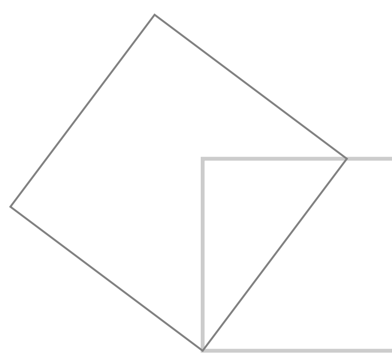
Addition of complex numbers is just vector addition. Multiplication is more interesting. The length of wz is the product $\|w\|\,\|z\|$, and the argument of wz is the sum of the arguments of w and z. Another way to say this is that muliplication by the complex number w rotates all complex number by $\arg(w)$, and scales lengths by $\|w\|$. This can be seen by trigonometry or linear algebra, since if $w = r\cos\theta + ir\sin\theta$ then

$$w(x+iy) = (rx\cos\theta - ry\sin\theta) + i(ry\sin\theta + rx\cos\theta)$$

or

$$\text{multiplication by } w\text{:}\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\ .$$

Exercise 1.1. *In the following figure, what w has been applied? What is its argument and length?*



Think of the complex numbers as a plane. Then any linear transformation is associated to a matrix, that whose columns are the images of $(1,0)$ and $(0,1)$. In our case multiplication by $w = a + bi$ takes $1 = (1,0)$ to $a + bi = (a,b)$, and $i = (0,1)$ to $-b + ai = (-b,a)$. Its matrix is therefore

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\ .$$

In fact, the matrices correspondng to multiplication by a complex number are exactly those of this form. In other words, *if we are given a matrix*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then it amounts to multiplication by a complex number when $a = d$ and $b = -c$.

2. The roots of simple equations

If z is a complex number, then z^n has as length the n-th power of $\|z\|$, and if θ is its argument then $n\theta$ is the argument of z^n. If $z^n = 1$ then $\|z\| = 1$ and $n\arg(z)$ must be the angle 0 (or equivalent to it). Therefore there are exactly n roots of this equation, namely all the numbers $\cos(2\pi k/n) + i\sin(2\pi k/n)$ with $k = 0, \ldots, k = n-1$.

Exercise 2.1. *Plot all the roots of $z^3 = 1$, $z^5 = 1$. Plot and write down explicitly all roots of $z^8 = 1$.*

Exercise 2.2. *Find all solutions of the differential equation $y'''' = y$.*

Exercise 2.3. *(a) Suppose $z^5 = 1$ but $z \neq 1$. Calculate*

$$z^2 + z^{-2} + z + z^{-1} + 1 \ .$$

(b) Use this to find a quadratic equation with integral coefficients satisfied by $z + z^{-1}$. (c) Find exact formulas, in terms of fractions and $\sqrt{5}$, for $\cos 72^\circ$ and $\sin 72^\circ$. (d) Find an exact formula for the ratio of the side of a regular pentagon to its radius.

Exercise 2.4. *Suppose $z^n = 1$ and $z \neq 1$. Calculate*

$$z^{n-1} + z^{n-2} + \cdots + z + 1 \ .$$

Exercise 2.5. *Let*

$$P(z) = z^3 - 3z + 1 \ .$$

Sketch carefully the image of the unit circle $\|z\| = 1$ under the map $z \mapsto P(z)$. Of the circles $\|z\| = r$ with $r = 0.25, 0.5, 2$ (all in the same figure). Find the roots of $P(z) = 0$ to 6 decimals in any way you can, and plot them on the same picture.

3. Complex differentiable functions

This course is concerned with calculus for **complex-valued valued functions of a complex variable**. Examples include all polynomials, also the exponential function e^z, as well as rational functions like $1/(1+z^2)$.

Exercise 3.1. *Define $\cos z$ and $\sin z$ for complex z. What is $\cos i$?*

It turns out that in order for calculus to make sense, we have to restrict our attention to only certain functions of this type, which are called **complex differentiable**. A function $f(z)$ is called differentiable if there exists another complex complex function $g(z)$ such that for small numbers Δz we have

$$f(z + \Delta z) = f(z) + g(z)\,\Delta z$$

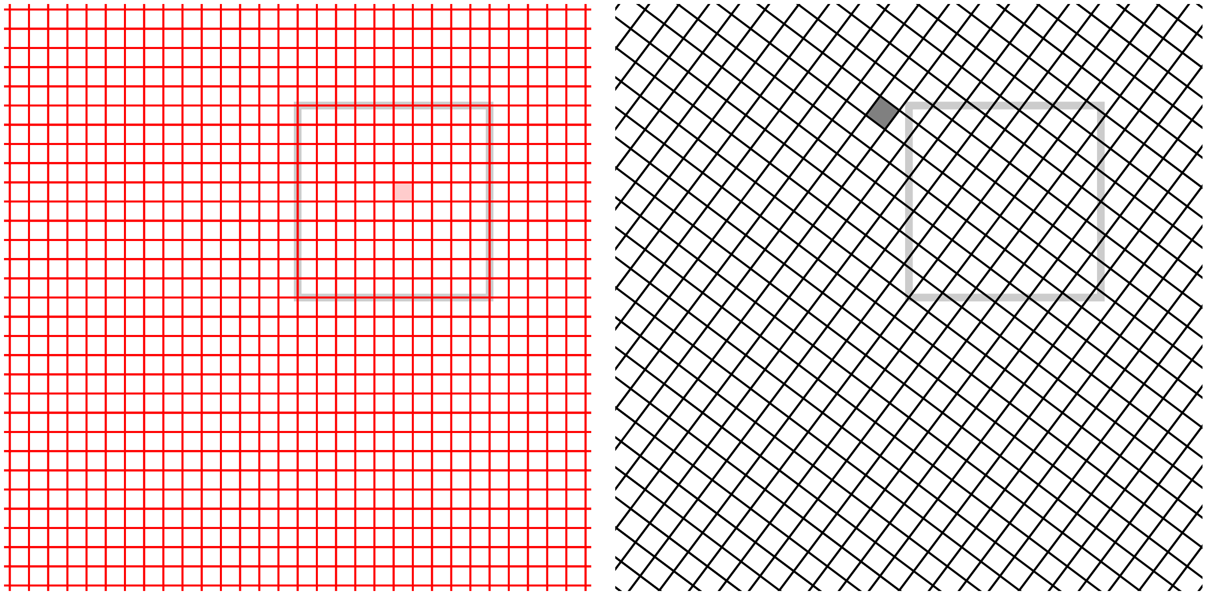
up to terms of order $(\Delta z)^2$ and higher, everywhere that $f(z)$ is defined. In this case $g(z)$ is said to be the derivative of $f(z)$, and expressed as $f'(z)$. Thus

$$(z + \Delta z)^2 = z^2 + 2z\,\Delta z + (\Delta z)^2$$

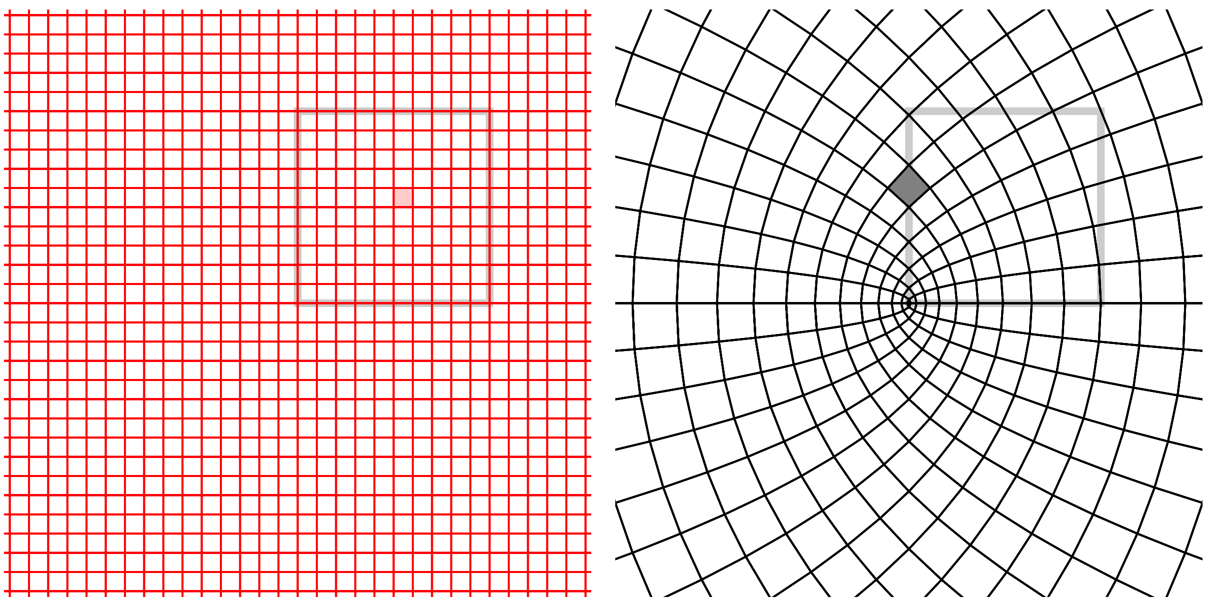
so that z^2 is complex differentiable with derivative $2z$.

Nearly all the functions you are familiar with such as polynomials or the exponential function are differentiable, and they have the obvious derivatives.

There is a geometric way to picture complex differentiable functions. The equation above says that the relative displacement Δz, when f is applied, becomes approximately the relative displacement $f'(z)\Delta z$. We know what multiplication by $f'(z)$ does—it scales and rotates. In particular it takes small squares into small squares. Thus a complex differentiable function is one which maps a grid of small squares into a grid, possibly curvy, of small squares. The simplest example is just multiplication by a fixed complex number w.



But more interesting is the function $f(z) = z^2$.



What happens near 0 is particularly interesting. Here the derivative $2z$ is 0, so we don't see any first order effects at all, but only second order ones. In general, first order effects are as we have described, but higher order ones can be quite complicated.

Exercise 3.2. *What is the approximate image of the square* $1 \le x \le 1.1$, $1 \le y \le 1.1$ *under the map* $f(z) = z^3 - 3z + 1$*? Sketch it. The image of the circle* $\|z\| = \epsilon$*, with* ϵ *small?*

Any complex-valued function of a complex variable can be thought of as a map which takes pairs of numbers (x, y) to other pairs of numbers (X, Y), where X and Y depend on x and y. For example, $z \mapsto z^2$ is the map

$$(x, y) \mapsto (x^2 - y^2, 2xy)$$

since $(x+iy)^2 = x^2 - y^2 + 2ixy$. If we are given a map in this real-coordinate form, how can we tell whether it is complex-differentiable or not? Suppose

$$f(x,y) = (X(x,y), Y(x,y)) \ .$$

Since (approximately)

$$\begin{aligned} X(x+\Delta x, y+\Delta y) &= X(x,y) + (\partial X/\partial x)\Delta x + (\partial X/\partial y)\Delta y \\ Y(x+\Delta x, y+\Delta y) &= Y(x,y) + (\partial Y/\partial x)\Delta x + (\partial Y/\partial y)\Delta y \end{aligned}$$

we also have approximately

$$\begin{aligned} f(x+\Delta x, y+\Delta y) &= (X(x+\Delta x, y+\Delta y), Y(x+\Delta x, y+\Delta y)) \\ &= (X(x,y) + (\partial X/\partial x)\Delta x + (\partial X/\partial y)\Delta y, Y(x,y) + (\partial Y/\partial x)\Delta x + (\partial Y/\partial y)\Delta y) \\ &= (X(x,y), Y(x,y)) + ((\partial X/\partial x)\Delta x + (\partial X/\partial y)\Delta y, (\partial Y/\partial x)\Delta x + (\partial Y/\partial y)\Delta y) \end{aligned}$$

or

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \begin{bmatrix} \partial X/\partial x & \partial X/\partial y \\ \partial Y/\partial x & \partial Y/\partial y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \ .$$

This will be complex differentiable just when the matrix is of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

or in other words when

$$\begin{aligned} \partial X/\partial x &= \ \ \partial Y/\partial y \\ \partial X/\partial y &= -\partial Y/\partial x \end{aligned}$$

Exercise 3.3. *Which of the following formulas come from complex differentiable functions?* (a) $(x^2+y^2, 2xy)$; (b) $(x,-y)$; (c) $(2xy, -x^2+y^2)$; (d) $(3x+2y, -x+y)$; (e) $(x,-y)/(x^2+y^2)$; (f) (y,x); (g) $(-y,x)$.