

A brief sketch of vector calculus

Chapter 1. Introduction to vector calculus

A **vector field** in n dimensions assigns an n -dimensional vector to each point of some region in n dimensions. Such things arise naturally in physics. For example, at any point in space a very small charged particle will feel an electrical force proportional to its charge. This force will be exerted in a certain direction, and the ratio of the strength of the force to the charge on the particle will be independent of this charge. Therefore we have at each point both direction and magnitude, hence a vector. This vector field is called the electric field, and is an example of a **force field**, where a vector measures force. Another example is the gravitational field. Yet another, slightly more complicated example is the magnetic field, where a vector measures the torque in space felt by a small magnet at any point. In this course, these are the most important examples, but they are hard to picture intuitively, and we shall also work with other examples. In a moving fluid, each particle in the fluid will have a certain velocity, which is also a vector, and if the flow itself is **stationary**, this velocity of a particle will depend only on the position of the particle (and not on the time at which it is observed). This vector field is called the velocity field of the flow, an example of a **geometric flow**. Another example in physics which we shall see occasionally is heat flow, where the vector field measures the transport of heat inside some substance. This will be called a **transport flow**. Yet another example of a vector field is purely mathematical in nature, the **gradient field** associated to any scalar-valued function.

1. Gradients

If $f(x, y)$ is a function of two variables x and y , then its gradient at (x, y) is the vector $[\partial f/\partial x, \partial f/\partial y]$. This notion can be defined also in one or three dimensions. In particular, if $f(x)$ is a function of one variable then its gradient is the one-dimensional vector whose component is $f'(x)$, or in other words the derivative of f . The gradient in higher dimensions is a natural generalization of that, enabling us to calculate the way in which f changes in various directions just as $f'(x)$ measures the change of f in the only direction it could.

Note that the coordinates of the position of a point are written as (x, y) , while coordinates of a vector, which measures relative position, are written $[x, y]$.

The gradient also has a natural intrinsic characterization, in that its components measure the rate of change of f in various directions, but in fact we can use it to measure rate of change in any direction. If $\Delta P = [\Delta x, \Delta y]$ is a small change from $P = (x, y)$ then we have **approximately**

$$f(P + \Delta P) = f(P) + \text{grad}(f) \bullet \Delta P = f(x, y) + (\partial f/\partial x)\Delta x + (\partial f/\partial y)\Delta y .$$

In other words, the gradient bundles in one package (a vector field) the general idea of rate of change of f .

2. The gradient of a vector field and related quantities

A vector field v in two dimensions has two components $[v_x, v_y]$. Each of these has a gradient associated to it, which gives 4 derivatives in all. The **gradient of the vector field** is then the matrix

$$\begin{bmatrix} \partial v_x/\partial x & \partial v_x/\partial y \\ \partial v_y/\partial x & \partial v_y/\partial y \end{bmatrix}$$

made up from these four components. Similarly in three dimensions for $v = [v_x, v_y, v_z]$ its gradient is the 3×3 matrix

$$\begin{bmatrix} \partial v_x/\partial x & \partial v_x/\partial y & \partial v_x/\partial z \\ \partial v_y/\partial x & \partial v_y/\partial y & \partial v_y/\partial z \\ \partial v_z/\partial x & \partial v_z/\partial y & \partial v_z/\partial z \end{bmatrix}$$

and in one dimension we just recover a 1×1 matrix whose entry is dv_x/dx .

This gradient matrix plays no direct role in most applications of vector fields. Instead, one uses some expressions related to it. The first is the **divergence** of v , which is defined in any number of dimensions to be the **trace** of the gradient matrix, the sum of its diagonal entries. Thus in three dimensions we have

$$\operatorname{div}(v) = \partial v_x / \partial x + \partial v_y / \partial y + \partial v_z / \partial z .$$

A matrix is called **symmetric** when the entries a_{ij} are the same as a_{ji} , or in other words when it is the same as the matrix we get by reflecting it along the NW-SE diagonal. We can decompose any matrix in any number of dimensions into two parts, its **symmetric** and **anti-symmetric** components. Since

$$\begin{aligned} x &= (1/2)(x+y) + (1/2)(x-y) \\ y &= (1/2)(x+y) - (1/2)(x-y) \end{aligned}$$

for any x and y we have

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= (1/2) \begin{bmatrix} 2a_{11} & (a_{12} + a_{21}) \\ (a_{12} + a_{21}) & 2a_{22} \end{bmatrix} \\ &+ (1/2) \begin{bmatrix} 0 & (a_{12} - a_{21}) \\ -(a_{12} - a_{21}) & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= (1/2) \begin{bmatrix} 2a_{11} & (a_{12} + a_{21}) & (a_{13} + a_{31}) \\ (a_{21} + a_{12}) & 2a_{22} & (a_{23} + a_{32}) \\ (a_{31} + a_{13}) & (a_{32} + a_{23}) & 2a_{33} \end{bmatrix} \\ &+ (1/2) \begin{bmatrix} 0 & (a_{12} - a_{21}) & (a_{13} - a_{31}) \\ -(a_{12} - a_{21}) & 0 & (a_{23} - a_{32}) \\ -(a_{13} - a_{31}) & -(a_{23} - a_{32}) & 0 \end{bmatrix} \end{aligned}$$

In each case, twice the second (anti-symmetric) component is called the **rotational component** of the matrix. We shall see later what it has to do with rotations. In two dimensions, an anti-symmetric matrix

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

is determined by a single number $-a$, and in three dimensions an anti-symmetric matrix

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

is determined by the three numbers a, b, c . We can assemble these into a vector

$$\begin{bmatrix} -c \\ b \\ -a \end{bmatrix} .$$

We shall see later why these particular choices of signs are made.

More explicitly, we get from a matrix (a_{ij}) in 2D the single number $-(a_{12} - a_{21})$, and in 3D the vector

$$\begin{bmatrix} -(a_{23} - a_{32}) \\ (a_{13} - a_{31}) \\ -(a_{12} - a_{21}) \end{bmatrix} .$$

If v is a vector field then the rotational component of its matrix gradient is called its **curl**.

Thus in two dimensions the curl of $[v_x, v_y]$ is

$$\partial v_y / \partial x - \partial v_x / \partial y$$

and in three dimensions the curl of $[v_x, v_y, v_z]$ is the vector

$$[\partial v_z / \partial y - \partial v_y / \partial z, -(\partial v_z / \partial x - \partial v_x / \partial z), \partial v_y / \partial x - \partial v_x / \partial y].$$

The best way to think of the curl in 2D is as a special case of curl in 3D for a vector field $[v_x, v_y, 0]$ which happens to have last component 0. Its 3D curl will have only a single non-vanishing component, equal to its 2D curl.

We shall see later what the significance of *div* and *curl* are, and give a geometric interpretation of them as well as of the matrix gradient. For now, you should just memorize the formulas for all these things. Well, that's not correct. What you should remember are procedures for recovering those formulas. There are some simple tricks. Let ∇ be the vector whose components are the partial derivative operators:

$$\nabla = [\partial / \partial x, \partial / \partial y, \partial / \partial z].$$

Then, at least formally, we have

$$\begin{aligned} \operatorname{div}(v) &= \nabla \cdot v \\ \operatorname{curl}(v) &= \nabla \times v. \end{aligned}$$

The calculation of curl will use the matrix

$$\begin{bmatrix} \partial / \partial x & \partial / \partial y & \partial / \partial z \\ v_x & v_y & v_z \end{bmatrix}.$$

by calculating the determinants of various 2×2 subdeterminants, and changing the sign of the middle term. In other words, we can write the curl of the vector field $[v_x, v_y, v_z]$ in terms of determinants as

$$\left[\begin{vmatrix} \partial / \partial y & \partial / \partial z \\ v_y & v_z \end{vmatrix}, - \begin{vmatrix} \partial / \partial x & \partial / \partial z \\ v_x & v_z \end{vmatrix}, \begin{vmatrix} \partial / \partial x & \partial / \partial y \\ v_x & v_y \end{vmatrix} \right].$$

Example. Let

$$v = [x, y].$$

Then its matrix gradient is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence its divergence is a constant 2, and its curl is identically 0.

Exercise 2.1. Calculate the matrix gradient, divergence, and curl for

$$v = [-y, x].$$

Exercise 2.2. Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a matrix in 2D, and let v be the vector field which associates $A[x, y]$ to the point (x, y) . Calculate its matrix gradient, divergence, and curl.

Exercise 2.3. Similar problem for a 3D matrix $A = (a_{ij})$.

Exercise 2.4. Calculate $\text{div}(\text{grad}(f))$ in $2D$, where $f = f(x, y)$ is a scalar field.

Exercise 2.5. Calculate $\text{curl}(\text{grad}(f))$ in $3D$, where $f = f(x, y, z)$ is a scalar field.

Exercise 2.6. Sketch the vector field $[-y, x]$.

Exercise 2.7. Sketch the vector field $[-x, -y]$.

Exercise 2.8. Sketch the vector field $v = [-y, -x]$. Calculate $\text{div}(v)$ and $\text{curl}(v)$.

Exercise 2.9. Sketch the vector field $[-y/(1+x^2+y^2), -x/(1+x^2+y^2)]$. Find its divergence and curl.

Exercise 2.10. Find a formula for the divergence and curl of the vector field

$$[-y, x]/f(r)$$

where $r = \sqrt{x^2 + y^2}$.

3. Maxwell's equations

Just as ordinary differential equations are often the natural way to describe the way finite physical systems evolve in time, partial differential equations are often the natural way to describe how fields evolve. For reasons that may become more apparent later on, these equations usually involve div , grad , and curl . The most famous case is the set of Maxwell's equations for electric and magnetic fields. Let \mathbb{E} be the electric field, and \mathbb{B} the magnetic field. Here are Maxwell's equations:

$$\begin{aligned}\text{div}(\mathbb{E}) &= \rho/\epsilon_0 \\ \text{curl}(\mathbb{E}) &= -\partial\mathbb{B}/\partial t \\ \text{curl}(\mathbb{B}) &= (1/c^2)(\partial\mathbb{E}/\partial t + j/\epsilon_0) \\ \text{div}(\mathbb{B}) &= 0\end{aligned}$$

Here j is the vector describing electric current density, ϵ_0 is a constant of proportionality depending on the choice of units, ρ is the charge density, and c is . . . the speed of light!

These equations look, and are, complicated. One of our principal goals is to investigate what these equations say about how the fields behave.

Exercise 3.1. Write down Maxwell's equations when the fields do not change with time.