Chapter 2. Introduction to flows

1. Review—matrices and linear transformations

If A is an $n \times n$ matrix and v an n-dimensional vector, then Av is another n-dimensional vector. The transformation taking v to Av is called linear. For example, in two dimensions we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} .$$

The matrix has a simple geometric interpretation. If we apply

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we get

$$\begin{bmatrix} a \\ c \end{bmatrix}$$

and if we apply it to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

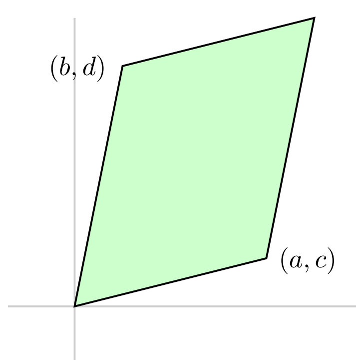
we get

$$\begin{bmatrix} b \\ d \end{bmatrix}$$

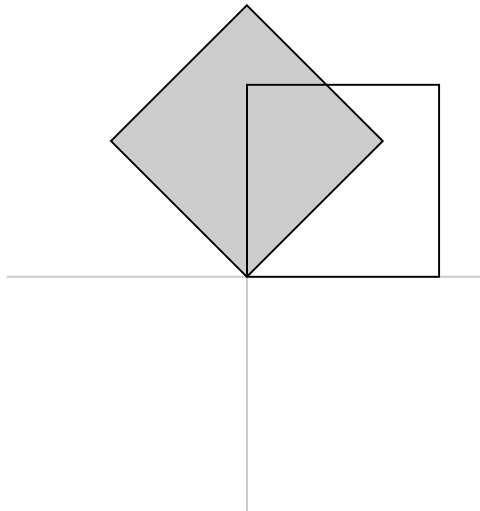
These images are the columns of A. Now the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are the edges of a unit square with lower left corner at the origin, aligned with the coordinate axes. So the columns of a matrix in 2D are the edges of the image of the square that the coordinate unit square gets transformed to by A. Doing this backwards, if we know what a linear transformation does to the coordinate square then we can read off the matrix.



Example. Suppose our transformation rotates points around the origin by 45°. Thus we have the following picture:



The edges of the new square are

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \qquad \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

so the matrix of the transformation is

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Similarly, rotation by t radians has matrix

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

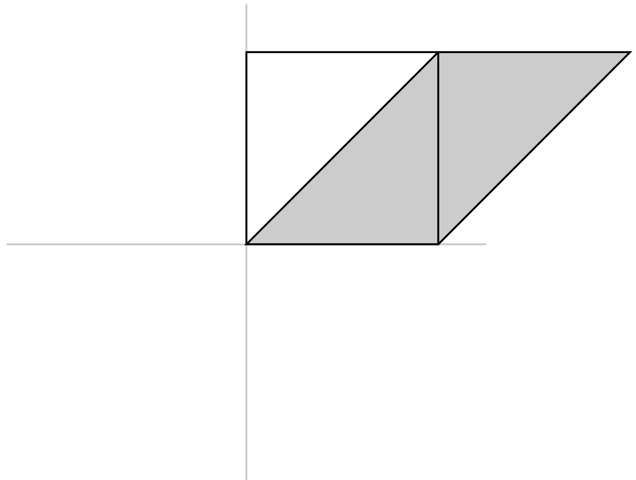
One basic fact is that if we first apply a linear transformations T and then a second S, with B the matrix of T and A that of S, the combined transformation has matrix AB.

Exercise 1.1. *What is the matrix corresponding to scaling in all directions by the constant c?*

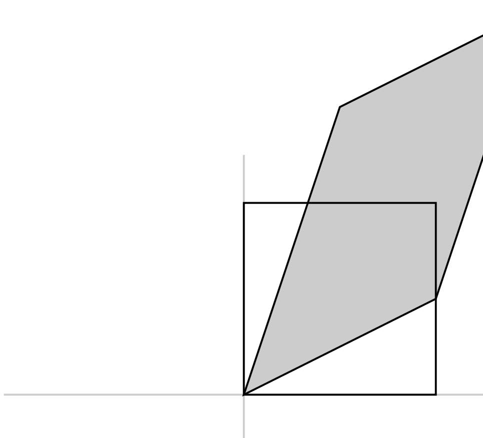
Exercise 1.2. *What is the matrix corresponding to rotation by 90°? 180°? 270°? 30°?*

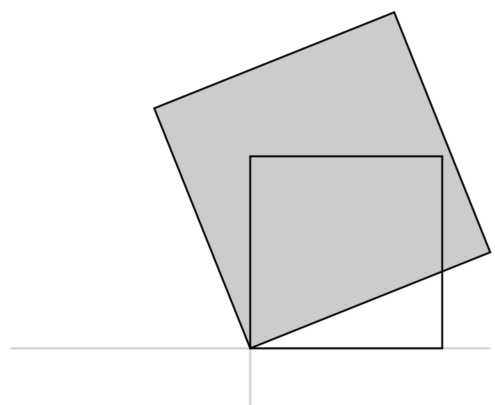
Exercise 1.3. *Suppose we first rotate by angle t and then scale in all directions by c. What is the final matrix? What do we get if we do these operations in the opposite order?*

Exercise 1.4. *What is a matrix corresponding to this picture? Find a second! Why are there two?*



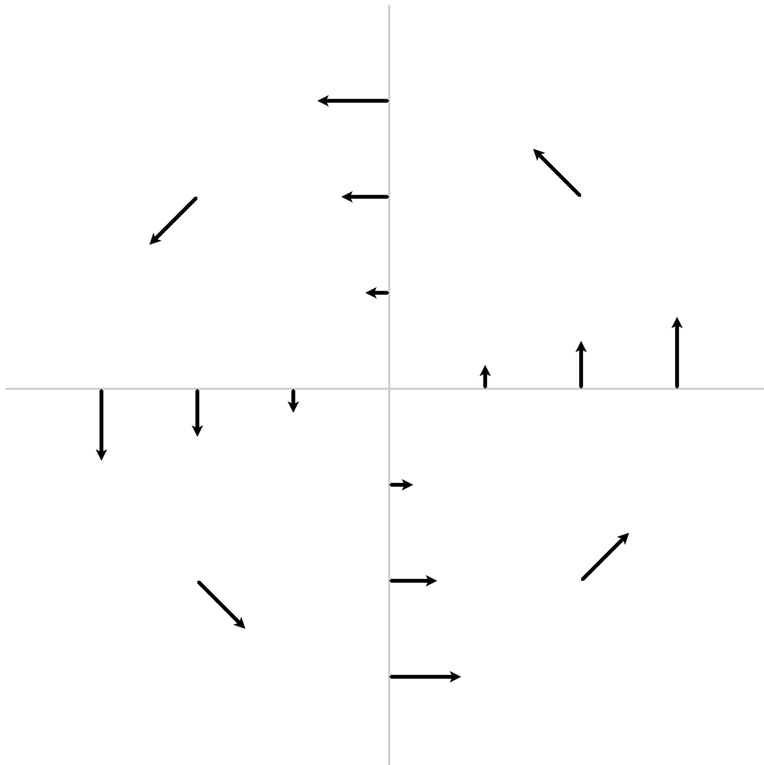
Exercise 1.5. *Same for these two pictures?*





Vector fields and velocity

Here is the picture of a vector field. At each point (x, y) we sketch an arrow corresponding to $v(x, y)$.



Exercise 1.6. *What is $v(x, y)$ here?*

The picture suggests motion. In fact, I think it is fair to say that it suggests a **flow** of the whole plane, which is to say it suggests that every point in the plane is in motion. In this case, all points have a uniform angular velocity, which implies that farther points have to move faster. The explicit interpretation here is the vector field tells us what the velocity at each point is.

In a flow, each point will be moved along as time proceeds. Suppose that point P is at position $F_t(P)$ at time t. Then its velocity at time $t = 0$ is

$$\lim_{h \to 0} \frac{F_h(P) - F_0(P)}{h}$$

The flow is called **stationary** if the velocity at each point does not depend on time. This doesn't mean that the points don't move, but just that **the pattern of the flow doesn't change in time**. In case of a stationary flow, the velocity at any point gives rise to a vector field.

Exercise 1.7. *If*

$$F_t(P) = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad P = \begin{bmatrix} x \\ y \end{bmatrix}$$

what is the vector field?