Chapter 4. More about flows

If we are given a vector field $v(x,y)$ with $v(x,y) = (v_x(x,y), v_y(x,y)))$ then finding the flow of a point $P = (x_0, y_0)$ means finding the location $F_t(x,y)$ of the particle P at any time t, where we are given that at time $t = 0$ it is located at P. The significance of the vector field is that it specifies what the velocity of the particle must be at any point in the flow. Fix $P = (x_0, y_0)$ for the moment, and let $F_t(x,y) = (x(t), y(t))$ be the location of P at time t. Finding what $x(t)$ and $y(t)$ are means neither more nor less than solving the pair of differential equations and initial conditions

$$\begin{aligned} x' &= v_x(x,y) \\ y' &= v_y(x,y) \\ x(0) &= x_0 \\ y(0) &= y_0 \ . \end{aligned}$$

Exercise 0.1. *Sketch the following vector field. In one word, what are the flow paths?*

$$v(x,y) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ .$$

Find the flow map for time t. Let S be the square with lower left corner at $(1,0)$, of side 0.25 aligned along the axes. Find the images of S in the flow at times 0, 0.5, 1, 1.5, 2. Draw them in your sketch.

Exercise 0.2. *Same problem for the vector field*

$$v(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ .$$

(You'll have to review from your course on differential equations.)

1. Some simple examples

Let's look at a simple example. Suppose

$$v = [ax, by]$$

where a and b are constants. The system of differential equations we get is

$$\begin{aligned} x' &= ax \\ y' &= by \\ x(0) &= x_0 \\ y(0) &= y_0 \end{aligned}$$

with solution

$$\begin{aligned} x &= x_0 e^{at} \\ y &= y_0 e^{bt} \end{aligned}$$

which we can write as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

In other words, the effect on a point in time Δt is to apply

$$\begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

to it.

This is in some respects similar to the flow with uniform angular velocity ω. In that case the vector field is $[-\omega y, \omega x]$, and the effect of the flow on a point is to apply

$$\begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

to it.

Both these examples illustrate a general phenomenon, that for a vector field $v = [ax + by, cx + dy]$ where

$$v(cx, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with a, b, c, d constant, the flow is of the form

$$F_t \begin{bmatrix} x \\ y \end{bmatrix} = E(t) \begin{bmatrix} x \\ y \end{bmatrix}$$

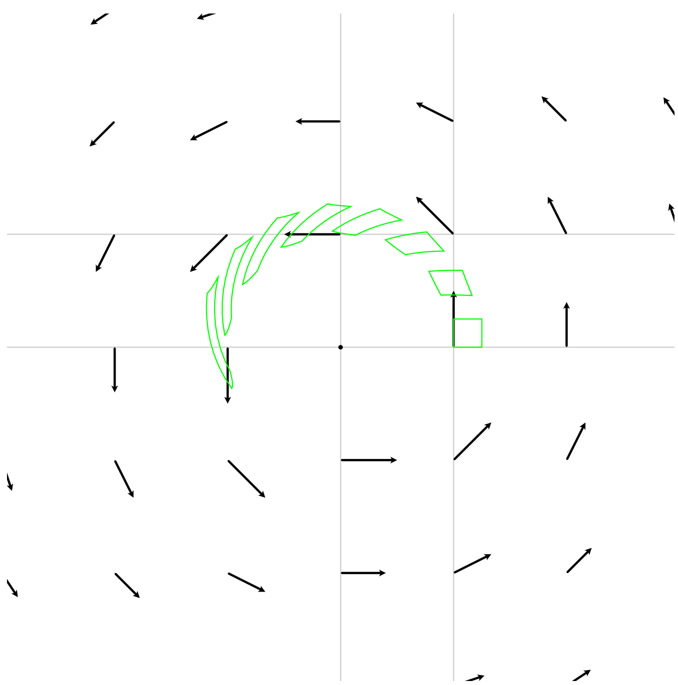
where E depends only on time and not position.

Exercise 1.1. *Find $E(t)$ if*

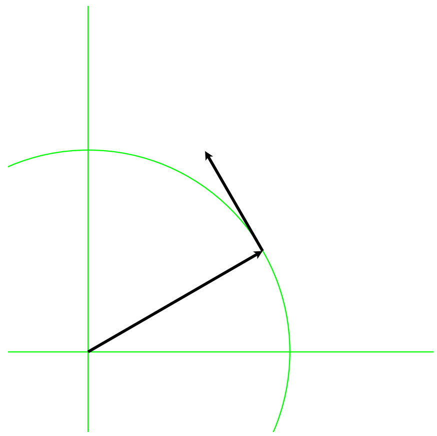
$$A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

2. A more complicated example

Let's look at another example. Here is one of the rotational vector fields from an earlier section:



The motion of any particle is in a circle around through the origin. The direction of the vector at (x, y) is therefore in the same direction as the radius vector rotated by $90°$.



This implies that the vector field is equal to a scalar times $[-y, x]$. You can see from the picture that the scalar depends only on the distance from (x, y) to the origin, and that it decreases as (x, y) goes towards infinity. In fact

$$v(x, y) = \frac{1}{1 + x^2 + y^2} [-y, x] \, .$$

So we are interested in solving a system of differential equations

$$x' = \frac{-y}{1 + x^2 + y^2}$$
$$y' = \frac{x}{1 + x^2 + y^2}$$

Looks messy. But a little thought will make things much easier. A particle on the circle of radius r moves around that circle with velocity $1/(1 + r^2)$. That means that in time t the point (x, y) is moved to

$$F_t(x, y) = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \omega = \frac{1}{1 + x^2 + y^2} \, .$$

Of course this formula is different from an earlier one because ω is not constant.

Exercise 2.1. *Consider the flow*

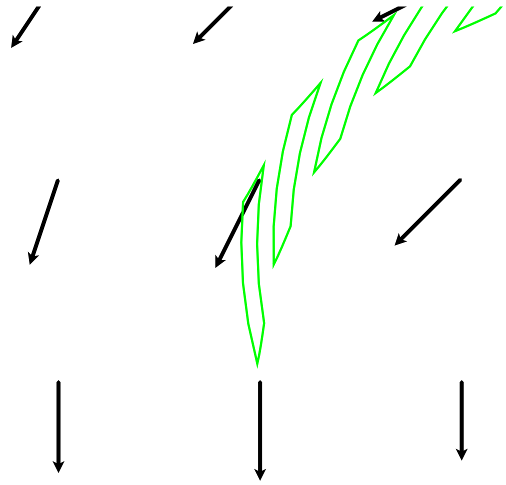
$$F_t(x, y) = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \omega = \frac{1}{1 + x^2 + y^2} \, .$$

Let S be the square of side 0.25, aligned along the axes, whose lower left corner is at $(2, 0)$. Find and sketch the image of S in this flow at $t = 1$, $t = 2$.

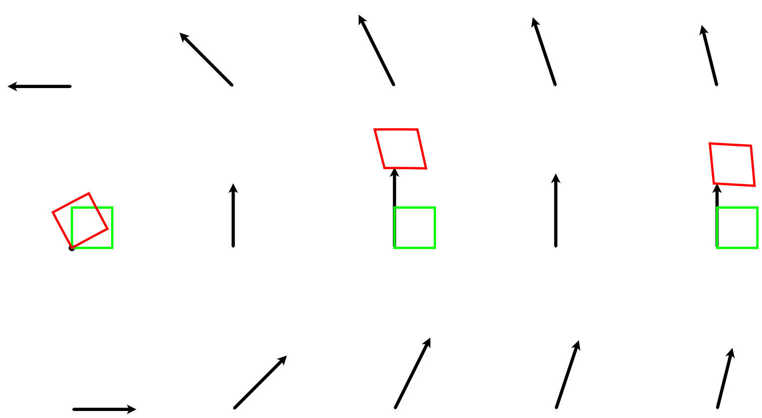
3. Relative motion in the flow

In this course our main concern is not with finding explicit formulas for the flow of a vector field, but with something technically much simpler. We are interested in *how small shapes change in the flow in the course of relatively short time intervals*. Another way of putting this is that we want to see what sort of instantaneous changes the flow is making.

What's different about short time intervals? The main difference is apparent in the figure of the rotational flow at the beginning of this chapter, and is made clearer if we enlarge part of that figure. If we put a small square in the flow at the point $(1, 0)$ and plot what it looks like at various multiples of a certain time interval, then after a while it gets distorted quite a bit, and in particular its sides are distorted into curves.



But if we look only at a short interval of time, the square doesn't get a chance to distort much. Although it certainly changes shape, the shape remains essentially linear.



In effect, in going from $t = 0$ to $t = \Delta t$ the square undergoes a transformation which is very closely the combination of (1) a translation and (2) a linear transformation. In the figure above, you can even see that this is true for $\Delta t = 1$, which is not particularly small. In moving from (x, y) in the flow after a time interval Δt, a first approximation to what happens to (x, y) is that it moves to $(x, y) + v(x, y)\,\Delta t$. But what happens to the square is not quite so simple. That is because each corner of the square gets translated by Δt times the velocity at that corner, but the velocities at the corners are all different (unless the vector field is constant). In other words, the square undergoes a basic translation, plus something else caused by the fact that the vector field changes from one point to another.

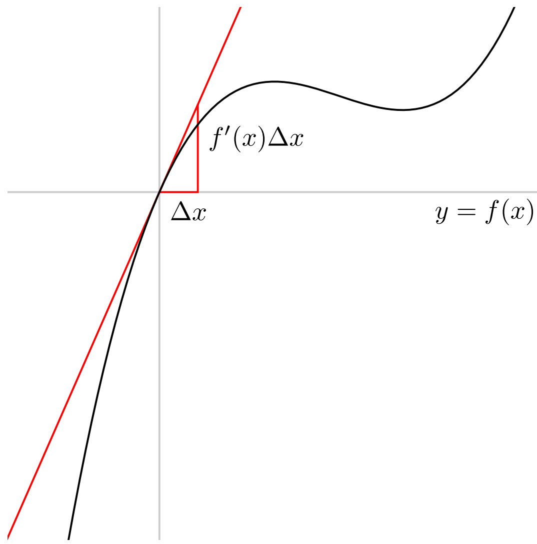
The next question is, *How do we take into account how the vector field changes?*

4. The vector field at neighbouring points

Here we will answer the question: *What sort of approximation can we make for the difference between vectors at nearby points?*

This is similar to questions you have seen before in other courses.

Example. Suppose $f(x)$ is a function of one variable. If Δx is small then the graph $y = f(x)$ is close to its tangent line near x.



This means that as a first order approximation we have

$$f(x + \Delta x) \approx f(x) + f'(x)\,\Delta x\ .$$

which is perhaps the single most important formula of calculus.

Example. Suppose $f(x, y)$ is a function of two variables. If Δx and Δy are small then the graph $z = f(x, y)$ is close to its tangent plane near (x, y). This means that we have the approximation

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + (\partial f/\partial x)\,\Delta x + (\partial f/\partial y)\,\Delta y$$

or

$$f(P + \Delta P) \approx f(P) + \mathrm{grad} f \bullet \Delta P$$

since

$$\mathrm{grad} f = [\partial f/\partial x, \partial f/\partial y]\ .$$

For a vector field, the idea is the same, but it just looks a bit more complicated at first. Suppose we want to compare the vector field at points $P = (x, y)$ and $P + \Delta P$ where $\Delta P = (\Delta x, \Delta y)$. We write

$$v(x + \Delta x, y + \Delta y) = [v_x(x + \Delta x, y + \Delta y), v_y(x + \Delta x, y + \Delta y)]$$

and to each of these coordinates we can apply the formula for a single function to get

$$v_x(x + \Delta x, y + \Delta y) \approx v_x(x, y) + (\partial v_x/\partial x)\Delta x + (\partial v_x/\partial y)\Delta y$$
$$v_y(x + \Delta x, y + \Delta y) \approx v_y(x, y) + (\partial v_y/\partial x)\Delta x + (\partial v_y/\partial y)\Delta y$$

which we can write as a single matrix equation:

$$v(P+\Delta P) \approx v(P) + \begin{bmatrix} \partial v_x/\partial x & \partial v_x/\partial y \\ \partial v_y/\partial x & \partial v_y/\partial y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \qquad \Delta P = [\Delta x, \Delta y] \ .$$

In other words, *the matrix gradient helps us calculate an approximation for the vector field in the neighbourhood of a point just as the derivative helps us approximate a function of one variable in the neighbourhood of a point.*

Exercise 4.1. *Find a formula for the matrix gradient of the vector field*

$$v(x,y) = \frac{[-y,x]}{1+r^2} \ .$$

Use the approximation formula above to find an approximate value for the vector field at $(1+\Delta x, \Delta y)$. *With* $\Delta x = 0, 0.2$ *and* $\Delta y = 0, 0.2$ *(four points in all), use what you find to draw approximately the vector field at the corners of a square* 0.2 *on a side, with lower left corner at* $(2,0)$. *Also find the matrix gradient at* $(0,0)$.

5. The flow in the neighbourhood of a point and over short time intervals

Over a short time interval Δt, what happens to a point P in a flow? Very roughly, it goes to

$$P_* = P + v(P)\Delta t \ .$$

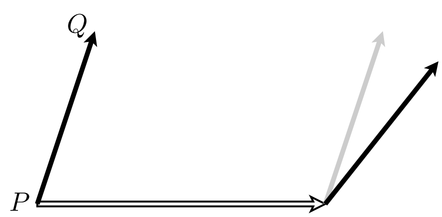
What happens to a point $Q = P + \Delta P$ near P? It goes approximately to

$$\begin{aligned} Q_* &= Q + v(Q)\Delta t \\ &= (P+\Delta P) + v(P+\Delta P)\Delta t \\ &\approx (P + v(P)\Delta t) + (\Delta P + M\Delta P\Delta t) \\ &= P_* + (I + M\Delta t)\Delta P \ . \end{aligned}$$

where M is the matrix gradient at P. In other words: The displacement of Q relative to P is ΔP. The relative displacement of Q_* relative to P_* is

$$(I + M\Delta t)\Delta P$$

So *to a very close approximation, the effect of the flow over small distances and small time intervals is to apply the matrix* $(I + M\Delta t)$ *to relative displacements* ΔP.



And we can read off the matrix gradient from the effect on squares:

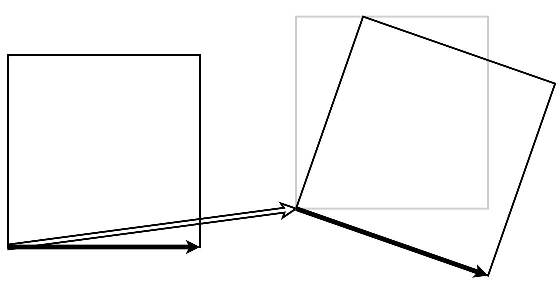


Figure 1

Exercise 5.1. *Assuming* $\Delta t = 1$ *in Figure 1, what is the matrix gradient?*

Exercise 5.2. *If* $v(x, y) = [-\omega y, \omega x]$ *what is* $(I + \mathrm{grad}(v)\Delta t)$*?*

Exercise 5.3. *If* $v(x, y) = [ay, bx]$ *what is* $(I + \mathrm{grad}(v)\Delta t)$*?*

Exercise 5.4. *If*

$$v(x, y) = \frac{[-y, x]}{1 + r^2}$$

what is $(I + \mathrm{grad}(v)\Delta t)$ *at the origin?*