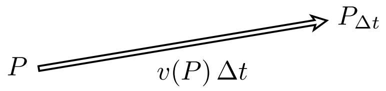
Chapter 5. The fundamental formula for flows

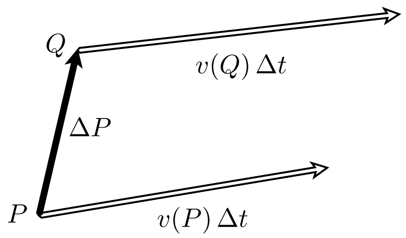
We shall discuss once again the fundamental formula relating the matrix gradient to the way shapes change in a flow.

1. The formula

Let me review the problem. We are trying to find a good approximation for what happens to a small shape in a flow over a short interval of time Δt. We fix a point P, and since displacement is equal to the product of velocity and time, we see that if $v(P)$ is the vector field at P then P moves approximately to $P_{\Delta t} = P + v(P)\Delta t$.



But now we want to see what happens to points nearby P. Say $Q = P + \Delta P$ where ΔP is small. Then Q is displaced by $v(Q)\Delta t$. This is not necessarily the same translation that P made, because in going from P to Q the vector field may have changed.

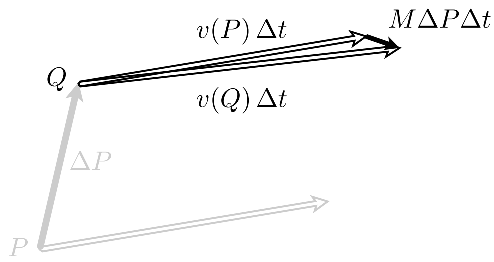


In order to estimate better where Q moves to, we use the approximation

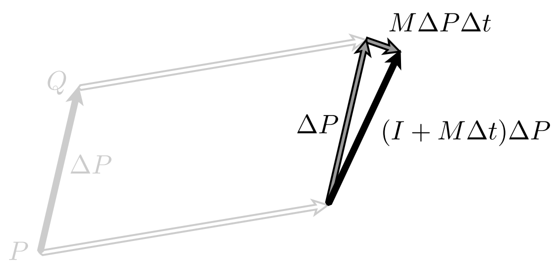
$$v(Q) = v(P + \Delta P) = v(P) + M(P)\Delta P$$

where $M(P)$ is the matrix gradient of the vector field at P. We multiply by Δt to get

$$v(Q)\Delta t = v(P + \Delta P)\Delta t = v(P)\Delta t + M(P)\Delta P \Delta t\ .$$



Finally, we can picture this in a different way:



which gives us the approximation we want:

- *If P flows to $P_{\Delta t}$ in time Δt, then a relative displacement of ΔP at P becomes a relative displacement of $(I+M\Delta t)\Delta P$ at $P_{\Delta t}$.*

2. Some examples

Rotation

Suppose $v(x,y) = [-\omega y, \omega x]$. Then v is the vector field of a constant angular rotation ω. The flow takes P to $R_t P$ where

$$R_t = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} .$$

The exact formula for what the flow does to a relative displacement of $\Delta P = [\Delta x, \Delta y]$ in time Δt is that it takes it to

$$\begin{bmatrix} \cos\omega\Delta t & -\sin\omega\Delta t \\ \sin\omega\Delta t & \cos\omega\Delta t \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} .$$

Since the matrix gradient of v is

$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

the approximation we get is

$$\begin{bmatrix} 1 & -\omega\Delta t \\ \omega\Delta t & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} .$$

The approximation is reasonable because

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$
$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

In effect, the approximation we are making is obtained by neglecting terms of degree two or higher in Δt.

Hyperbolic flow

Here we take

$$v(x,y) = [-x/4, y/4] .$$

The flow takes $P = [x,y]$ to

$$\begin{bmatrix} e^{-t/4}x \\ e^{t/4}y \end{bmatrix} = \begin{bmatrix} e^{-t/4} & 0 \\ 0 & e^{t/4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

The matrix gradient is

$$M = \begin{bmatrix} -1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$$

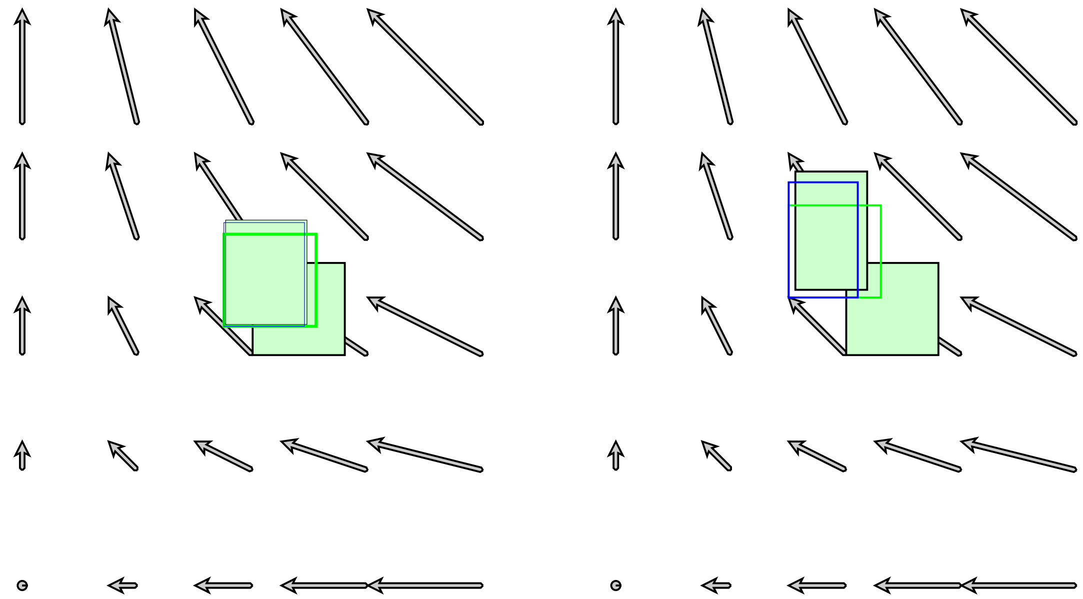
and

$$I + M\Delta t = \begin{bmatrix} 1 - \Delta t/4 & 0 \\ 0 & 1 + \Delta t/4 \end{bmatrix} .$$

Here we are making the approximation

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

In this last case, we have these pictures, where we show a square; its true image in the flow after various times Δt, its translation by $v\Delta t$, and the first order approximation we get from the formula involving the matrix gradient.



3. Determinants, areas, and volumes

In the next section we need to recall from your course on linear algebra that in any number of dimensions, the linear transformation associated to a matrix M changes all areas (volumes) by scaling them by the determinant of the matrix. Thus

- *The linear transformation whose matrix is*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

scales all areas by the factor $(ad - bc)$, taking into account how it changes orientation.

This follows from a straightforward geometric argument for the unit coordinate square, and it follows from all other regions because they are made up of limits of squares translated and scaled from this one.

If the matrix is

$$M = \begin{bmatrix} 1 + hm_{1,1} & hm_{1,2} \\ hm_{2,1} & 1 + hm_{2,2} \end{bmatrix}$$

then its determinant is

$$1 + h(m_{1,1} + m_{2,2}) + h^2(m_{1,1}m_{2,2} - m_{1,2}m_{2,1}) = 1 + hT + h^2 D$$

where T is the trace of the matrix M and D its determinant. If h is small then h^2 is very small, and this is close to $1 + hT$.

4. The meaning of divergence

We know that in an interval of time Δt, a flow changes small shapes near a point P by applying the linear transformation $(I + M\Delta t)$ where M is the matrix gradient of the vector field at P. This can be written in 2D as

$$\begin{bmatrix} 1 + (\partial v_x/\partial x)\Delta t & (\partial v_x/\partial y)\Delta t \\ (\partial v_y/\partial x)\Delta t & 1 + (\partial v_y/\partial y)\Delta t \end{bmatrix}$$

When Δt is small then its determinant is essentially

$$1 + (\partial v_x/\partial x + \partial v_y/\partial y)\Delta t$$

which is also

$$1 + \mathrm{div}(v)\Delta t \ .$$

Thus, in a short interval Δt, an area A is changed into an area roughly $(1 + \mathrm{div}(v)\Delta t)A$. The difference between the two is

$$\mathrm{div}(v)A\Delta t$$

and therefore the rate of change at time $t = 0$ is

$$\mathrm{div}(v)A \ .$$

Another way of saying this is that

- *In a flow, the divergence at a point is the relative rate of change of area (volume) in the flow near that point.*

This result is exactly the reason the divergence is important. One consequence is that the sign of the divergence determines whether areas (volumes) are increasing or shrinking. In particular, if the divergence is 0 everywhere in a vector field, then the area of any object remains constant in the flow. This is very nearly what happens in real water flows, for example, because water is almost incompressible.