Chapter 6. How large areas and volumes change in a flow

The question we now want to address is this: Let R be any region in the plane where a vector field v is given. In contrast to the previous problem, R doesn't have to be small. *What is the rate of change of the area of R in the flow (at time $t = 0$)?*
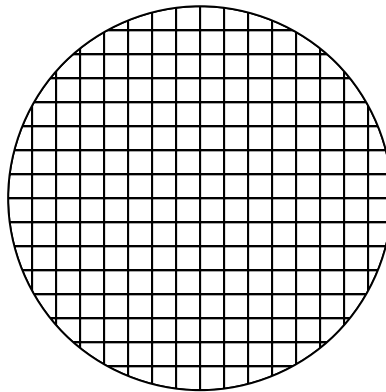
1. The integral formula

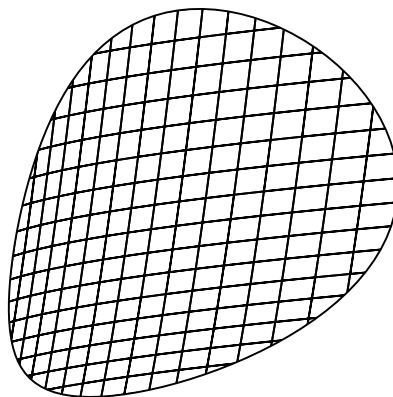The area of R is the double integral

$$\int_R dA$$

which is the limit of sums

$$\sum \Delta A$$

where the finite regions ΔA get smaller and smaller.

After a time interval Δt each area ΔA changes in the flow to $(1 + \mathrm{div}(\mathrm{v})\Delta t)\Delta A$.

Thus the entire area of R changes, to a close approximation, to

$$\int_R (1 + \text{div}(\text{v})\Delta t)\, dA \ .$$

The difference between the new area and the old is, up to first order in Δt,

$$\text{difference} \ = \ \Delta t \int_R \text{div}(\text{v})dA \ .$$

Therefore, the rate of change of area at $t = 0$ is the limit ratio of the difference to Δt, which is

$$\int_R \text{div}(\text{v})dA \ .$$

In other words, the answer to our question is this:

- *The rate of change of the area of a 2D region R in the flow at $t = 0$ is the double integral*

$$\int_R \text{div}(\text{v})dA \ .$$

The same argument works in 3D to give a rate of change of volume.

- *The rate of change of the area of a 3D region R in the flow at $t = 0$ is the triple integral*

$$\int_R \text{div}(\text{v})dV \ .$$

Exercise 1.1. *Let R be the square in 2D with corners at $(\pm 1, \pm 1)$. If v is the flow $v(x, y) = [ax, by]$, what is the rate of change of the area of R in the flow at $t = 0$?*

Exercise 1.2. *In the previous problem, find an exact formula for the area of the transformation of R at time t.*

Exercise 1.3. *If R is the same square, but*

$$v(x, y) = [ax + cy, bx + dy]$$

find (a) the rate of change of the area of R at $t = 0$; (b) an exact formula for the area of the transformation of R at time t. (Hint: find a simple differential equation for the area at time t.)

Exercise 1.4. *If R is the cube with corners $(\pm 1, \pm 1, \pm 1)$ and*

$$v(x, y, z) = [a_{1,1}x + a_{1,2}y + a_{1,3}z, a_{2,1}x + a_{2,2}y + a_{2,3}z, a_{3,1}x + a_{3,2}y + a_{3,3}z]$$

find an exact formula for the volume of the transformation of R at time t. (Hint: find a simple differential equation for the volume at time t, using the divergence of v.)

2. The boundary formula

There is another way to calculate the change of area. Let C be the boundary of R. Then C changes in the flow, and the total change of area of the region can be measured just by examining how C moves. Where C moves in, area decreases, and where C moves out, it increases.
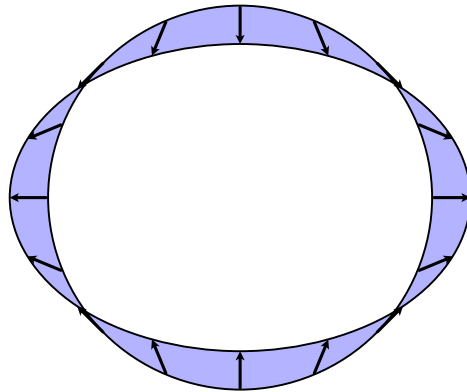
In order to find out how the area changes, we just have to track what happens to the boundary.

To be precise, what we have to do is find out how pieces of the boundary are swept out in the flow. A segment of length Δs along C is essentially just translated in the flow, by roughly the vector $v_P \Delta t$ if P is a point in the middle of the segment. It therefore sweeps out roughly a parallelogram whose sides are the segment and the vector $v_P \Delta t$.
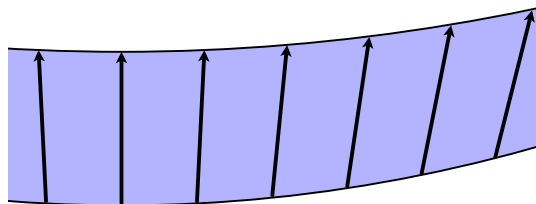
The area it sweeps out is the product of base by height. The base is Δs, and the height is the component of $v_P \Delta t$ perpendicular to the curve. We can even keep track of in and out motion if we make the height negative on the inward side. But this (signed) perpendicular component is the dot product of $v_P \Delta t$ with the unit normal **unit normal vector n** to the curve at P.
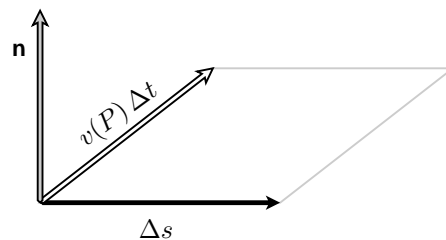
The total area swept out by the whole curve, and the change in area of the region, is therefore

$$\Delta t \int_C v(x, y) \bullet \mathbf{n} \, ds$$

and the rate of area change is

$$\int_C v(x, y) \bullet \mathbf{n} \, ds$$

where we integrate in the counter-clockwise direction to get the signs right.

The two formulas we get for the rate of area change must agree. Therefore we get **Gauss' Theorem**:

$$\int_R \mathrm{div}(v) \, dA = \int_C v \bullet \mathbf{n} \, ds \ .$$

In dimension one, a vector field is just a scalar function $[f(x)]$. The divergence is $[f'(x)]$. The endpoints a and b are moving in the flow at velocities $f(a)$ and $f(b)$. The rate at which the length is changing therefore $f(b) - f(a)$. The formula above becomes

$$\int_a^b f'(x) \, dx = f(b) - f(a) \ .$$

and Gauss' Theorem is a kind of generalization of this.

3. Flux in 3D

In 3D Gauss' Theorem becomes:

- *If R is a 3D region and S its boundary surface, then*

$$\int_R \mathrm{div}(v)(x, y, z) \, dx \, dy \, dz = \int_S v \bullet \mathbf{n} \, dA$$

The volume of R is changing because stuff is flowing through its boundary, and the integral is called the **flux** integral of v over S.

In both 2D and 3D Gauss' Theorem is largely a theoretical tool, since none of the integrals can be calculated easily except in special cases. Here is a simple example. Suppose $v(x, y)$ is the vector field $[x, y]$ where $r = \sqrt{x^2 + y^2}$. Let R be the inside of the unit circle around the origin. Then v points straight out at each point on the boundary of R. Thus $v \bullet \mathbf{n}$ is just the constant 1, and we have

$$\int_C v \bullet \mathbf{n} \, ds = 2\pi \ .$$

Exercise 3.1. *Suppose v is the 2D vector field $[x, y]/r^n$. What is the flux integral of v over the circle of radius R centred at the origin?*

Exercise 3.2. *Suppose v is the 3D vector field $[x, y, z]/r^n$ in 3D. What is the flux integral of v over the sphere of radius R centred at the origin?*

4. The first of Maxwell's equations

Suppose we locate a point charge of magnitude μ at the origin. At any other point in space, according to Coulomb's Law, the electric field is inversely proportional to the square of the distance from the charge, and proportional to the charge. In other words

$$E = \frac{\kappa\mu[x, y, z]}{r^3}$$

for some constant κ depending on the choice of units.

Exercise 4.1. *Calculate the divergence of E at a point where $r \neq 0$.*

Let S be a sphere of radius R. Then on S the field vector E and **n** both point straight out. Therefore $E \bullet \mathbf{n}$ is equal to $|E|$, and the integral of flux is just the product of $4\pi R^2$ and $\kappa\mu/R^2$, which is equal to $4\pi\kappa\mu$, and in particular is independent of the radius. How can we explain this simple fact? You can calculate that the divergence of E vanishes everywhere except at the origin. Let R be the region between two spheres of radii R_0 and R_1. IOn the inside sphere, the normal vector out of R points inwards, and on the outside it points outwards from the origin. Therefore the integral over the boundary of R is the difference

$$\int_{r=R_1} E \bullet \mathbf{n} \, dA - \int_{r=R_1} E \bullet \mathbf{n} \, dA = \int_R \mathrm{div}(\mathrm{v}) \, dV = 0$$

which means precisely that the two fluxes are the same.
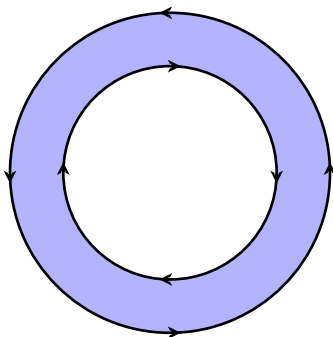
In fact, the same calculation can give us two rather broad generalizations.
- *If S is any single surface enclosing the charged particle, then the flux of E through S is $4\pi\kappa\mu$.*
- *If S is any surface not enclosing the particle, then the flux through S is 0.*

We can make up almost any electric field by assembling very very small charges. The field resulting from this collection of charges will be the sum of all the fields from the different bits. If S is any single surface containing all the charge, then the flux through S will be the sum of all the fluxes from the bits, so that the total flux will still be $4\pi\kappa\mu$ if μ is the total charge. If S contians some but not all of the charge, then the result is more interesting. The flux through S contributed by the charges outside S will be 0, and the total flux through S will be $4\pi\kappa\mu_S$ if μ_S is the charge contained inside S.

Suppose we take S to be a very small surface around a single point P. The flux through S will be $4\pi\kappa\mu_S$ on the one hand, and on the other, by Gauss' Theorem, it will be equal to to the integral of the divergence of E over the inside of S. This will be essentially the same as the product of volume vol_S and the divergence of E at P. Therefore $4\pi\kappa\mu_S = \mathrm{div}(\mathrm{E})\mathrm{vol}_S$ or $\mathrm{div}(\mathrm{E}) = 4\pi\kappa(\mu_S/\mathrm{vol}_S)$ or

- *Maxwell's first equation:*

$$\mathrm{div}(\mathrm{E}) = 4\pi\kappa\rho$$

   where ρ is the charge density at P.

There is no analogue of electric charge for a magnetic field—i.e. there is no such thing as an isolated positive magnetic pole. Maxwell's fourth equation formalizes this as

- *If B is the magnetic field, then*

$$\text{div(B)} = 0$$

  everywhere.

5. Technical remarks on the flux integrals

As has been remarked, it is not easy nor usually necessary to calculate explicit flux integrals, in either 2D or 3D. Nonetheless it is a good idea to have a recipe for doing so—just in case.

In 2D: We are give the region R in the plane, with boundary C, and a vector field $v(x, y)$. The boundary curve C might be made up of several pieces. For each one, find a **parametrization**—i.e. a map $t \mapsto (x(t), y(t))$ covering the curve as t goes from t_0 to t_1. Then at any point the vector field v is $v(x(t), y(t))$. The tangent vector is $(x'(t), y'(t))$, and the normal vector **n** is the rotation of this by $-90°$, if we are traversing the curve so the outside is to the right, and divided by its length $\sqrt{dx^2 + dy^2}$. This is $[y'(t), -x'(t)]$. the distance ds along the curve is $\sqrt{dx^2 + dy^2}$. The two square roots cancel out (thank goodness).

The flux of v through C is then

$$\int_{t_0}^{t_1} v(x(t), y(t)) \bullet [y'(t), -x'(t)] \, dt \ .$$

Exercise 5.1. *If $v = [-y, -x]$ and R is the region of points at distance less than 1 from the point $(1, 1)$, what is the flux through the boundary of R?*

In 3D: Here $(s, t) \mapsto (x(s, t), y(s, t), z(s, t))$ will be a parametrization of S. The vectors

$$v_s = [\partial x/\partial s, \partial y/\partial s, \partial z/\partial s]$$
$$v_t = [\partial x/\partial t, \partial y/\partial t, \partial z/\partial t]$$

will be velocity vectors of the parametrization, pointing along the lines $t = \text{const}$ and $s = \text{const}$, respectively. If the orientation of s and t is correct then the normal vector will be the normalization of $v_s \times v_t$. Again the square-roots involved in this normalization and the formula for area cancel out, and the flux is

$$\int_S v(x, y, z) \bullet (v_s \times v_t) \, ds \, dt \ .$$

This assumes that we have chosen s and t in accord with the right hand rule (which makes the cross product point out).

Exercise 5.2. *Let S be the doughnut of radius 1 around a circle of radius 2 centred at the origin in the plane $z = 0$. Find a parametrization for that circle. Find a parametrization for S. Find a formula for v_s and v_t, and if $v = [x, y, z]$ find a formula for the flux of v through S.*