Chapter 7. Curl

Understanding the geometric meaning of the curl of a vector field is more complicated than understanding its divergence. Very roughly, the curl of a vector field at a point measures the rotation taking place in the flow at that point. But this local behaviour turns out to be somewhat paradoxical, and it really can't be interpreted in a simple fashion. The notion of circulation around paths in the flow is required to make a clear statement possible.

1. Rotational motion in 3D

Uniform rotational flow around the origin in 2D is takes

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and has velocity field

$$v(x, y) = [-\omega y, \omega x] = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

The matrix gradient here is anti-symmetric. On the other hand, if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is any anti-symmetric matrix then $a = -a$, $d = -d$, and $b = -c$, so that the matrix is in fact

$$\begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}$$

and corresponds to a uniform rotational flow of angular velocity c. We can summarize:

- *In 2D, rotational flows of uniform angular velocity around the origin are those flows fixing the origin whose matrix gradients are constant anti-symmetric matrices.*

The same turns out to be true in 3D.

Uniform rotational motion in 3D takes place around a fixed axis. We represent it by a vector $[a, b, c]$ whose direction lies along the axis and whose magnitude is the angular velocity of the rotation. The angular velocity is measured positively according to the right hand rule. The basic formula, proven in physics courses, is that if P is a point anywhere in 3D and O any point on the axis, then the linear velocity (i.e. the real velocity, as opposed to the angular velocity) of P in the rotational motion is

$$v(P) = [a, b, c] \times \overrightarrow{OP} .$$

This determines the vector field of the flow in terms of the cross product with $[a, b, c]$.

Exercise 1.1. *Explain why $[a, b, c] \times \overrightarrow{OP}$ doesn't depend on the choice of O, as long as it is on the axis.*

Exercise 1.2. *In the rotational flow around the axis through the origin in the direction of $[1, 1, 1]$ at an angular velocity of π radians per unit of time, what are the velocities of the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$? At an angular velocity of ω?*

This formula can be translated into one involving matrices. We have

$$[a, b, c] \times [x, y, z] = [bz - cy, cx - az, ay - bx] = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

Since the map taking u to $v \times u$ is clearly linear, there must be some matrix effecting the transformation. What is important here is that this matrix is anti-symmetric. Furthermore, any symmetric matrix must be one of these, associated to a 3D vector $[a, b, c]$. In other words

- *In 3D, rotational flows of uniform angular velocity around an axis through the origin are those flows fixing the origin whose matrix gradients are constant anti-symmetric matrices, or equivalently those associated by means of the cross product to 3D vectors.*

We have calculated the vector field of uniform rotational flow, but we have not calculated the flow itself. This is a bit more difficult. To find the flow here we must answer the following question:

Exercise 1.3. *Suppose we rotate a point $P = (x, y, z)$ through an angle θ around the oriented axis through the origin in the direction of to the vector α. How do we calculate the rotated point?*

Answer this question for the case where $\alpha = [1, 1, 1]$ and $\theta = 90°$, and P is each of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. (Hint: The direction of α doesn't change if we divide it by its length to get a vector of length 1. This will simplify things. Think about the cross product.)

Exercise 1.4. *Answer this question with a formula for the case where $\alpha = [1, 1, 1]$, θ is arbitrary, and P is each of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.*

Exercise 1.5. *Let v be the rotational vector field in 3D corresponding to $\alpha = [1, 1, 1]$ and velocity ω. Look at the velocities on the plane $z = 0$, and at each point P of this plane obtain a vector field on the plane by projecting $v(P)$ onto the plane. Sketch this vector field. Try to decide whether or not it is a rotational flow. If it is, find a formula for its angular velocity. If it isn't, explain why not.*

2. Rotation in the flow

Let v be an arbitrary vector field, say in 2D for the moment. If M is its matrix gradient, then we can write M as the sum of its symmetric and anti-symmetric components. The curl of v is twice the anti-symmetric component, identified as a matrix rather than a vector. The anti-symmetric component of M is the rotational component in the following sense. In a short time Δt the point P moves to $P_{\Delta t} = P + v_P \Delta t$, and the point $P + \Delta P$ moves to

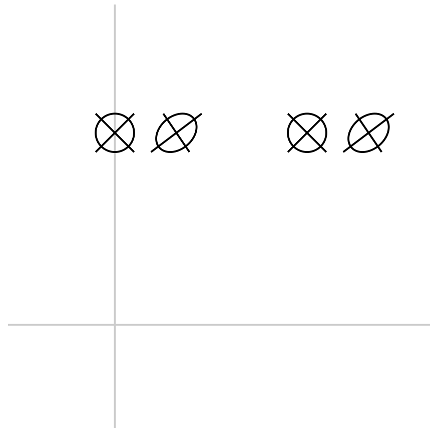
$$P_{\Delta t} + (I + M\Delta t)\Delta P$$

so the relative motion of points near P is described by the matrix $(I + M\Delta t)$, at least up to terms of first order in Δt. But if $M = S + A$ is the decomposition into symmetric and anti-symmetric components, then

$$(I + M\Delta t) = (I + S\Delta t + A\Delta t) = (I + A\Delta t)(I + S\Delta t) - AS(\Delta t)^2 = (I + A\Delta t)(I + S\Delta t)$$

if we don't worry about terms of order two in Δt. This means that the relative motion of points near P is the composition of transformations $(I + S\Delta t)$ and $(I + A\Delta t)$. The second is, essentially, a rotational motion. The first is, by results from linear algebra, a scaling transformation with respect to two orthogonal axes (determined by the eigenvectors of S). For example, a small circle will be transformed into an ellipse by the first and then rotated by the second. This seems not too difficult to understand. If the vector field is that of a uniform rotational flow, it matches exactly what we see. If the vector field is $[ax, by]$ we have a parabolic or hyperbolic flow and again the effect of the flow is just a compression or expansion in directions parallel to the coordinate axes. (Look back at the pictures in an earlier chapter.)

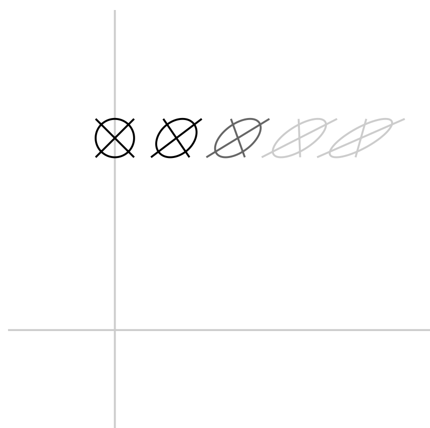
But in the mixed case, where the matrix gradient is neither symmetric nor anti-symmetric, what happens is more complicated. Let's look at a shear, for example. In the following picture we see what happens to a circle at various places in the flow over a periods of time. Looks good.



The effect we see is because the matrix gradient of this vector field canbe expressed

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = (1/2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (1/2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

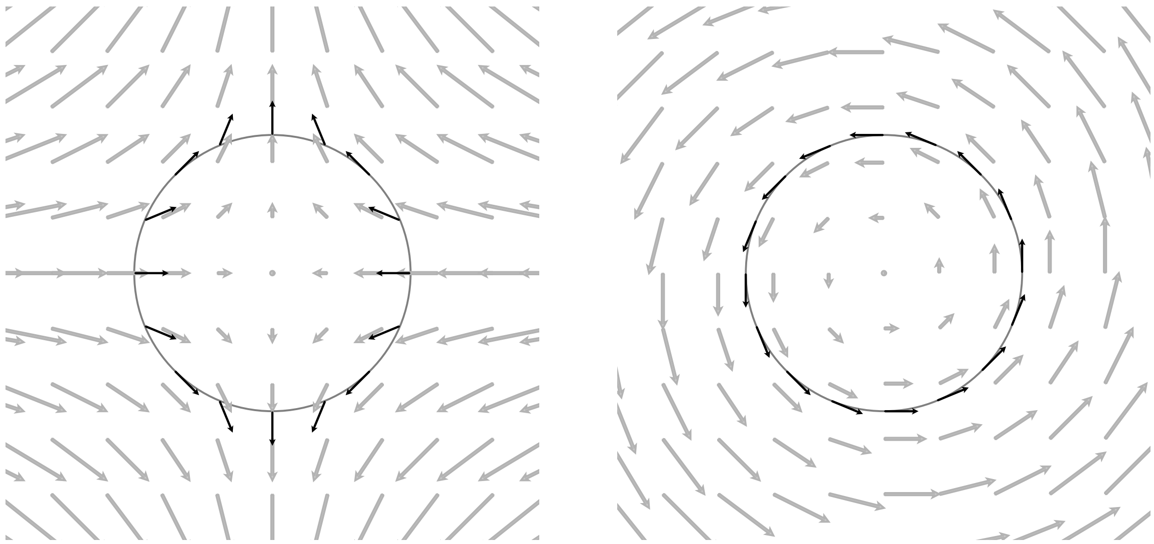
so that its curl, for example is everywhere equal to 1, and there is some rotational motion in the flow at every point in the plane. But this doesn't seem right, because we know exactly what the effect of a shear is, over any length of time. All those little rotations don't add up to a big one.



In other words, a shear looks like the combination of a rotation and scaling over any short period of time, but in the long run we don't see this effect any more. Definitely, things are potentially confusing. It seems hard to formulate anything very useful.

3. Circulation

Therefore we ask the question: *What long range effects are created by the curl of a vector field?* We know from the discussion above that the geometry of the flow doen't do. Let's look at two examples to see if we can get an idea of how things go. We will look at the two simplest flows again, $v = [-cy, cx]$ for some small value of c, and $v = [-ax, ay]$ for some small positive number a. The idea in each case is simple: we follow a circle counterclockwise around the origin and try to keep track of the general direction of the vector field as we do so, *compared to the direction we are traveling in.*



In the first case, the tangential component of v oscillates back and forth as we go all around the circle, while in the second it stays fixed. We can even see that in the first case the average tangential velocity is 0, while in the second it is positive (and proportional to the angular velocity the flow). Of course the average tangential velocity will be the integral around the circle of the dot product of v and the tangent vector, or in other words some constant times the integral

$$\frac{1}{2\pi R}\int_{C_R} v(P(t))\bullet \mathbf{t}\, ds$$

where **t** is the unit tangent vector at $P(t)$ along the circle C_R of radius R, where

$$P(t) = (R\cos t, R\sin t)\ .$$

The integral by itself is called the **circulation** of v around the circle. If the flow is that of uniform angular velocity ω the circulation around C_R is $2\pi R^2\omega$.

Exercise 3.1. *Calculate the circulation of the vector field* $[ax, by]$ *around* C_R.

More generally, the circulation of v along any path $(x(t), y(t))$ which starts at $t = t_0$ and ends at T_1 is defined to be

$$\int_{t_0}^{t_1} v(x(t), y(t))\bullet [x'(t), y'(t)]\, dt$$

It turns out to be independent of the parametrization and just depends on the geometrical curve the path follows.

Note that this integral is somewhat like the one defining the flux of v through the path, except that **t** replaces **n**.

The main fact regarding circulation is this (Stokes' Theorem in 2D):

- *If v is any vector field in the plane, then the circulation around any simple closed curve C equal to the integral of the curl of v over the inside of C.*

We have seen this in several examples. This theorem is important in going from Faraday's formulation of the laws of electricity and magnetism to that of Maxwell, and in particular in explining why light, radar, radio, etc. are all waves in the electromagnetic field. The circulation of the electric field in a closed loop of conducting wire is directly related to the current produced in the wire by the field, and also to the rate of change of the magnetic field. This phenomenon is related in turn to Maxwell's equation

$$\mathrm{curl}\ \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Exercise 3.2. *Find the circulation of the vector field* $[ax + by, cx + dy]$ *around the circle of radius* R*. Around the square with corners at* $(\pm R, \pm R)$*.*

In 3D things are slightly more complicated. Let S be any surface in 3D with a simple boundary C, assume S and C oriented compatibly according to the right hand rule. If v is any vector field defined around S, then the result we want is that

$$\int_S (\mathrm{curl}(v) \bullet \mathbf{n})\, dA = \int_C v \bullet \mathbf{t}\, ds$$

where **n** is at each point of S the normal vector pointing outwards from S according to the ritght hand rule. In other words, the left hand side is the flux of $\mathrm{curl}(v)$ through S that we have seen already in Gauss' Theorem. One striking feature of this result is that several choices of S are possible, but the integral on the left turns out to be independent of that choice.

Exercise 3.3. *Calculate* $\mathrm{div}(\mathrm{curl}(v))$ *for any vector field* v*. Explain roughly why the integral above doesn't depend on the choice of* S*? (Start: Suppose* S_1 *and* S_2 *were the top and bottom halves of a sphere, and* C *were the equator. Use Gauss' Theorem and the previous result.)*