

Chapter 8. Maxwell's equations and vector calculus

Most of the laws of electricity and magnetism, as well as the relationship between the two, were stated first in comprehensive form by the English experimental physicist Michael Faraday in the early nineteenth century. His formulation was, as we shall see, in terms of the way more or less real objects behaved, such as charges you could see and wires you could move—and also in terms of something he called the electric and magnetic fields, which you could only visualize and measure incompletely. Shortly afterwards, James Clerk Maxwell set down Faraday's laws in purely mathematical terms—partial differential equations which described the instantaneous and infinitesimal changes of state of the electric and magnetic fields. In so doing, he modified Faraday's laws in a significant way, and discovered the relationship between electricity, magnetism, and light. Above all, the formulations of Faraday and Maxwell led people to replace the older concept of action at a distance by that of the purely local action and transmission of physical phenomena through changing fields. It was one of the most important scientific achievements of the century, in a century noted for such things. In this Chapter we shall try to understand, through vector calculus, the transition from the laws discovered by Faraday and his predecessors to those of Maxwell.

1. Charge and current

The first thing to be understood is the mathematical way to say that current represents the flow of charge, and to express the law of conservation of charge.

Current is the flow of charge, and it can be represented by a vector field. This is an example of a new interpretation of vector fields in terms of **transport** of some kind of quantity, which is somewhat different from the **geometric** interpretation of vector fields in terms of the velocities in a flow which we have seen so far. The idea is this: in space we have something being transported around in a continuous way. For example, charge in a conducting surface or heat in a solid with an uneven temperature distribution. At any point, we want the vector field to measure the strength of the flow. What we do is put a very small surface into space, and then in a certain small time interval Δt measure the amount of stuff that flows through the bit of surface in that time. What is measured depends on the position of the surface relative to the flow. For example, if the flow is parallel to the surface nothing will be transported across it, and for a given area the amount will at a maximum when the surface lies perpendicular to the flow. In fact, there will at each point a vector v which measures the transport across small areas of size ΔA over small intervals of time in the sense that the flow through ΔA in time Δt will be

$$(v \cdot \mathbf{n}) \Delta A \Delta t .$$

In this interpretation, the flux through any surface, calculated by the integral

$$\int \int_S (v \cdot \mathbf{n}) dA$$

where \mathbf{n} is the unit normal to S , is the rate of flow of stuff through S .

Current J is in this sense the vector field associated to the flow of charge. Since charge in the universe at large is conserved, it cannot just suddenly disappear. Thus, if the amount of charge inside some region changes, it is because there is a current through the boundary of the region. In other words, the flux through the boundary must measure the rate of charge on the inside.

$$-d/dt \int \int \int_R \rho dV = \int \int \int_R \frac{\partial \rho}{\partial t} dV = \int \int_S (J \cdot \mathbf{n}) dA$$

where ρ is charge density inside R . The sign agrees with what we are saying, because an outward current means a decrease of charge inside. By Gauss' Theorem we know also that

$$\int \int_S (J \cdot \mathbf{n}) dA = \int \int \int_R \text{div}(J) dV .$$

Since this holds even for very small regions, where $\text{div}(J)$ is essentially constant, we have

$$\text{div}(J) = -\frac{\partial \rho}{\partial t}.$$

This is the mathematical formulation of two facts: (1) current measures the flow of charge and (2) charge is never destroyed, but only moved around from place to place. The second of these laws is not quite the same as saying that the total amount of charge is constant, because it rules out the possibility that it disappears suddenly at one place and reappears at another. In this, it is consistent with the general principle that in nature instantaneous action at a distance does not occur. This equation is at any rate often called the **equation of local charge conservation**, even though it doesn't say that charge distribution doesn't change, but only accounts for what change there is in terms of current, which is a measurable quantity.

2. The two systems of electric units

Before we go on to discuss other mathematical equations governing the way in which the electromagnetic field behaves, we should take a look at one of the more remarkable and subtle facts in this field. There are essentially two ways in which electricity creates forces on objects. The first is from static charge. If we put two small similar charges μ_i at a distance r from each other, then they will repel each other with a force proportional to the amount of charge and inversely proportional to the square of the distance between them. If we arbitrarily select some given size of charge as the unit, we will have Coulomb's Law

$$F = \kappa_e \frac{\mu_1 \mu_2}{r^2}$$

where κ_e is some constant depending on the units of mass, length, and time chosen. The 'e' here stands for 'electrostatic'. Choosing the units of charge and time also determine a unit of current, which measures charge per unit of time. It is probably worth mentioning here that, following Maxwell's locality principle, this force is due not directly to the action at a distance of one charge on another, but to the local action of the electric fields created by each of the charges.

The second is from steady currents. If we run a current through each of two parallel wires at distance r from each other, then there will be a force F per unit length of wire exerted on them. The force will be attractive if the currents flow in the same direction, and is given by the formula

$$F = -\kappa_m \frac{I_1 I_2}{r}$$

for some constant κ_m . The 'm' here stands for 'magnetic', because the force is actually caused by the magnetic field created by the currents. Note that these constants κ_e and κ_m are, in principle, perfectly reasonable things to determine experimentally—although in practice it was not easy to do accurately. One of the most remarkable facts discovered in the course of the nineteenth century is that

- *The ratio κ_e/κ_m is equal to $c^2/2$, where c is the speed of light.*

This coincidence was, I believe, noticed first (about 1856) by the German physicist Kirchhoff, the same one for whom the laws of circuits are named.

3. Charge and the electric field

As we have seen already, Coulomb's law allows us to calculate that the flux of the electric field through a closed sphere of radius R containing a charge μ is equal to $4\pi\kappa_e\mu$, and by applying Gauss' Theorem to deduce that

$$\text{div}(E) = 4\pi\kappa_e\rho$$

where ρ is charge density. There is one great advantage to this new version of Coulomb's law—Coulomb's law itself applies only to static fields and charge distributions. If a charged particle moves around, for example,

calculating what the electric field is must take into account the finite speed with which electrical effects travel. The new law, however, is completely local; there are no travel effects to take into account. It is valid for all electric fields at all times.

4. There are no isolated magnetic charges

The same reasoning, together with the fact that there are no isolated magnetic charges, gives us

$$\operatorname{div}(B) = 0$$

where B is the magnetic field.

5. Magnets in motion create currents

If you move a magnet through a loop of wire, it will cause a current. Currents are caused by electric fields, and can depend only on the direction of the electric field in the direction along the wire. Furthermore, it is caused by the accumulation of electric forces on each piece of the wire. Thus in a closed wire, we expect the circulation of the electric field in the wire to depend on the way in which the magnetic field inside the loop changes. How do we measure this change? By integrating the rate of change of the magnetic field over any area spanning the loop, or in other words by the flux of the magnetic field through that loop. This means

$$\int_C (E \cdot \mathbf{t}) ds = -d/dt \int \int_S (B \cdot \mathbf{n}) dA .$$

Now Stokes' Theorem tells us that the circulation of E is also equal to

$$\int \int_S (\operatorname{curl}(E) \cdot \mathbf{n}) dA$$

and since S can be very small and face in an arbitrary direction we see that

$$\operatorname{curl}(E) = - \frac{\partial B}{\partial t} .$$

6. Steady currents create magnetic fields

A steady current in a straight wire which is not depositing charge at any point produces a constant magnetic field circulating around the wire, proportional to the current strength and inversely to the distance from the wire. This means that the circulation of B in any circle around the wire will be independent of the circle's radius. The circulation of B through any loop is, by Stokes' Theorem, the flux of the curl of B through the inside of the loop. This suggests that the curl of B is always proportional to the current. If we are careful about units, we see that in any electric field with a steady current and charge distribution

$$\operatorname{curl}(B) = 4\pi \text{constant} J$$

for a suitable constant.

7. The displacement current

This is not the whole story. If the charge distribution is not steady, then an extra term has to be added to the last equation. This was first realized by Maxwell, and an explanation (which is a bit tricky) can be found in any physics text. The final result is that in all circumstances

$$\operatorname{curl}(B) = \text{constant}_1 \frac{\partial E}{\partial t} + \text{constant}_2 4\pi J .$$

for suitable constants.

8. Electromagnetic waves

The occurrence of the speed of light in comparing electric and magnetic fields is no accident. It has as one consequence that the effects of changes in an electric field are not instantaneous, but travel with speed c . For example, if a static charge were suddenly to begin to move, the effects would be felt at a distance r only in time r/c . This coincidence also suggested to Maxwell what we now know to be true, namely that light itself is a traveling disturbance in the electromagnetic field. This conjecture was verified only after his death, as was the existence of other types of electromagnetic radiation.