

Mathematics 266 — Spring 1999 — Part II

Chapter 2. Roots of polynomials

Complex numbers are introduced in order that every real number, whether negative or not, have a square root. But in fact every complex number has a square root. Namely, if z has magnitude r and argument θ , then the complex number with magnitude \sqrt{r} and argument $\theta/2$ will be a square root of z . The negative of this will be another. This implies in turn that any quadratic polynomial

$$aZ^2 + bZ + c = 0$$

has two roots, at least if we count multiple roots twice. What this means is that if z_1 and z_2 are roots then

$$aZ^2 + bZ + c = (Z - z_1)(z - z_2)$$

and if they are equal we say it has multiplicity two.

Exercise 0.1. Find the square roots of $3 + i$.

Exercise 0.2. Find the cube roots of $3 + i$.

Exercise 0.3. Find the roots of $Z^2 - iZ + 1 = 0$.

1. Why roots exist

What is much less easy to understand is why any complex polynomial

$$P(Z) = a_n Z^n + \cdots + a_0$$

has n roots in the same sense. Since we can always factor out $Z - z$ if z is a root, it suffices to see that $P(Z)$ has at least one root. We will not analyze why in detail, but let's look at one example, the cubic polynomial

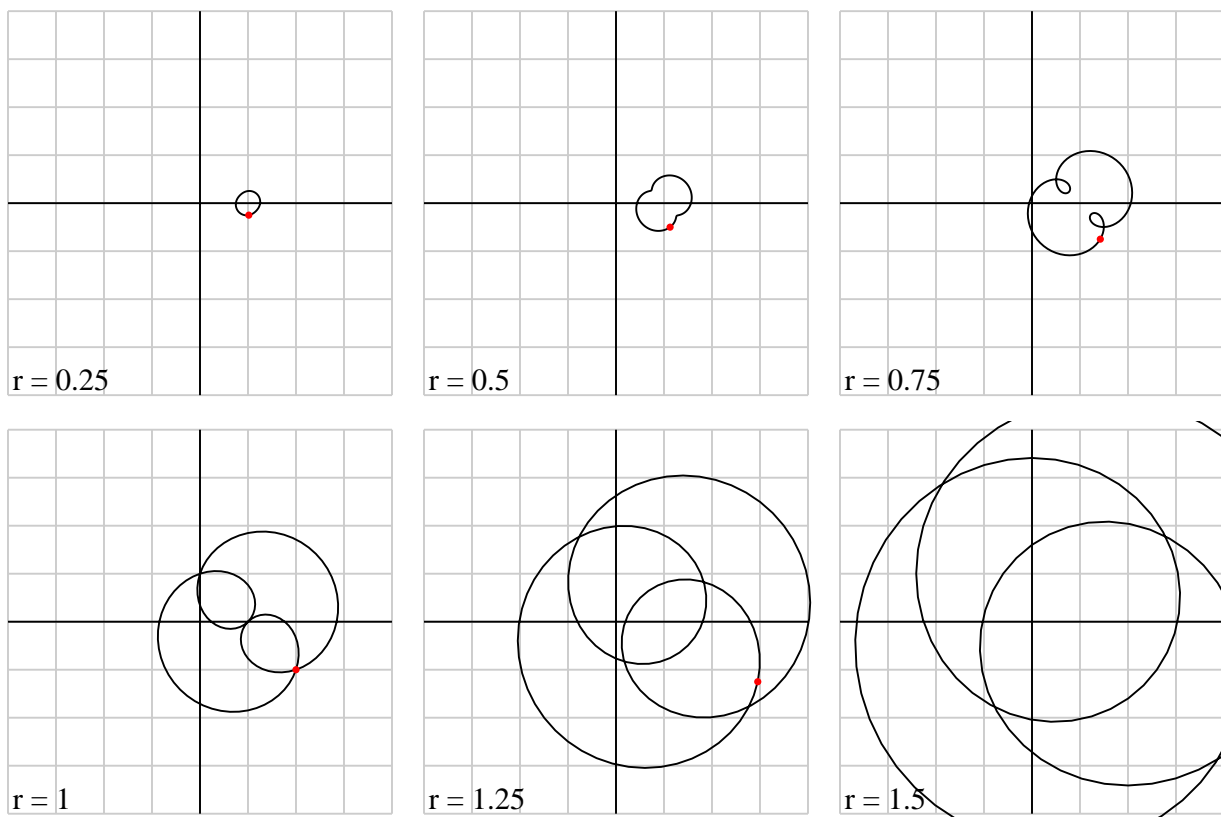
$$P(Z) = Z^3 - iZ + 1.$$

We do this geometrically, by considering the images under P of circles of increasing radii centred at 0 and starting with radius 0. Of course the circle of radius 0 is a single point, just $P(0) = 1$. It is invisible. The next radius we look at is $r = 0.25$. This is still pretty small, and $P(Z)$ is a complex differentiable function, so that

$$P(\Delta z) \sim P(0) + P'(0)\Delta z$$

for Δz of magnitude 0.25. Since $P'(0) = -i$, multiplication by $P'(0)$ scales by 1 and rotates by -90° , so the image is approximately a circle of radius 0.25 centred at 1.

The following pictures show successive images, which are not even close to circles for $r \geq 0.5$ (the small dot locates $P(r)$).



What you can see from these pictures is that as r gets larger and larger, the image of $\|z\| = r$ crosses through 0. When it does this, it means that on this circle there is a root of magnitude r . Note that it does this three times, meaning we can see, in some sense, how roots appear. We can also see that one root has magnitude close to 0.75, the next has magnitude a bit more than 1, and the third has magnitude a bit more than 1.25. We can also estimate roughly what the arguments are, since the argument at the dot is 0.

For an arbitrary polynomial $P(z)$, we may factor out powers of z if necessary to get $P(0) \neq 0$. Thus the images will start out as small circles around $P(0)$. They will therefore wrap around 0 zero times. For large z the value of $P(z)$ will be dominated by the z^n term, so has to wrap around the origin n times. In between, the images have to pass through 0, which will guarantee that a root exists.

2. How to find roots

It is not easy to find roots by computation. One simple idea is to start with an initial guess z_0 and apply Newton's method, getting successive approximations

$$z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)}.$$

This converges rapidly, as long as z_0 is close enough. It is tricky, however, to find a good starting guess. Probably easiest is to use geometry as suggested by the pictures above.

For

$$z^3 - iz + 1$$

if we start with $z_0 = 0$ we get successively

$$\langle 0.0000 \quad -1.0000 \rangle$$

$$\langle 0.1000 \quad -0.7000 \rangle$$

$$\langle 0.2313 \quad -0.6753 \rangle$$

$$\langle 0.2243 \quad -0.6903 \rangle$$

$$\langle 0.2245 \quad -0.6903 \rangle$$

$$\langle 0.2245 \quad -0.6903 \rangle$$

where $\langle x \ y \rangle$ is computer output for $x + iy$.

Exercise 2.1. Find the other roots of

$$z^3 - iz + 1 = 0$$

to the same accuracy.

Exercise 2.2. Find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$