Mathematics 266 — Spring 1999 — Part II

Chapter 5. Power series expansions

This chapter will explain how to extend Cauchy's integral formulas to allow for more complicated singularities.

1. Isolated singularities and Laurent series

Suppose that $f(z)$ is a complex differentiable function defined all around some point z_0, but not necessarily at z_0 itself. A typical example would be a negative power $1/(z-z_0)^n$ with $n > 0$. Then two things can happen: either $f(z)$ is in fact complex differentiable at z_0, or something goes wrong at z_0 and $f(z)$ is said to have an **isolated singularity** at z_0. In either case, we can consider the restriction of $f(z)$ to a circle $|z - z_0| = r$ for small values of r. As long as r is small enough, $f(z)$ will be defined and well behaved on that circle. If we parametrize the circle by $z(t) = z_0 + re^{it}$ then f has period 2π in t and we can expand $f(z)$ in its complex Fourier series

$$f(re^{it}) = \sum_{-\infty}^{\infty} c_{r,n} e^{int}$$

where the coefficients $c_{r,n}$ will depend on r and are given explicitly, according to the general theory of Fourier series, by the formula

$$c_{r,n} = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) e^{-int}\, dt\ .$$

In fact, we can express the coefficients $c_{r,n}$ in terms of a complex integral

$$c_{r,n} = \frac{r^n}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz\ .$$

This is because if we parametrize the circle as $z = z_0 + re^{it}$ so that $dz = ire^{it} dt$ we express the complex integral as

$$\begin{aligned} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz &= \int_0^{2\pi} f(re^{it}) \frac{ire^{it}}{r^{n+1} e^{i(n+1)t}}\, dt \\ &= \int_0^{2\pi} f(re^{it}) \frac{i}{r^n e^{int}}\, dt \\ &= \frac{2\pi i\, c_{r,n}}{r^n}\ . \end{aligned}$$

By Cauchy's First Integral Formula the complex integral

$$c_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

does not depend on r! So we can now write the Fourier series as

$$\begin{aligned} f(z) &= \sum_{-\infty}^{\infty} c_{r,n} e^{int} \\ &= \sum_{-\infty}^{\infty} c_n r^n e^{int} \\ &= \sum_{-\infty}^{\infty} c_n z^n\ . \end{aligned}$$

This is called the Laurent series of $f(z)$ at z_0. If $f(z)$ has no singularities then this series has only non-negative terms and becomes the **Taylor series**

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

If the Laurent series has only a finite number of negative powers, then $f(z)$ is said to have a **finite pole** at z_0. If The only negative power is $1/(z - z_0)$, then the pole is called **simple**. If f has a pole of order N at z_0 then

$$f(z) = \frac{F(z)}{(z - z_0)^N}$$

with $F(z_0) \neq 0$.

Exercise 1.1. *Describe all numbers z such that $e^z = 1$; $e^z = -1$; $e^z = i$.*

Exercise 1.2. *(a) Find all the places where*

$$\cos z = 0 \ .$$

Give solid reasons for your answer. (b) Tell the order of each zero. (c) Find the first three terms of the Taylor series at those points. (d) Find the first three terms of the Laurent series of $\tan z$ at all of its singularities. (e) Tell what the residues at those places are, too.

2. The Third Integral Formula

The main application of Laurent series is to calculate integrals of functions around singularities. Formally this is very simple. We know that for $n \neq -1$

$$\int_{z_1}^{z_2} (z - z_0)^n \, dz = \left[\frac{(z - z_0)^{n+1}}{n + 1} \right]_{z_1}^{z_2}$$

where z_1 and z_2 are points other than z_0. In particular if C is a closed curve not passing through z_0 then

$$\int_C (z - z_0)^n \, dz = 0$$

even if n is negative, as long as it is not -1. Therefore we can also calculate

$$\int_C f(z)\, dz = \int_C \sum_{-\infty}^{\infty} c_n z^n \, dz = \sum_{-\infty}^{\infty} c_n \int_C z^n \, dz = c_{-1} \int_C \frac{1}{z - z_0} \, dz = 2\pi i c_{-1}$$

if C winds around z_0 once counter-clockwise. In other words, if C is a curve containing only the single isolated singularity z_0 of f, then the integral

$$\int_C f(z)\, dz$$

depends only on the single coefficient c_{-1} in the Laurent series of f at z_0. This coefficient is called the **residue** of $f(z)$ at z_0.

More generally, if C contains isolated singularities $z_1, z_2, \ldots$ and C winds around them each once counterclockwise, then

$$\int_C f(z)\, dz$$

is equal to $2\pi i$ times the sum of the residues at the singular points z_n.

3. Calculating the residue

Let's look at a relatively simple example. Say

$$f(z) = \frac{1}{(z^2-1)^2}$$

and C is the curve going from $-2i$ around a semi-circle to $2i$, then straight down to $-2i$. The only singularity inside C is $z=1$, so we have to calculate the residue of $f(z)$ at 1. The calculation we do here will be fairly typical.

Step 1. We first write

$$\frac{1}{(z^2-1)^2} = \frac{1}{(z+1)^2}\frac{1}{(z-1)^2} = \frac{F(z)}{(z-1)^2}$$

much as we would have to apply Cauchy's Second Formula.

Step 2. We write down the first few terms of the Taylor series at z_0 of the 'good factor' $F(z)$. I recall that the Taylor series of a function $F(z)$ at z_0 is

$$F(z) = \sum_0^\infty c_n (z-z_0)^n$$

where the coefficients can be evaluated in terms of the derivatives of $F(z)$ at z_0:

$$c_n = \frac{F^{(n)}(z_0)}{n!}$$

and in particular

$$c_0 = F(z_0), \qquad c_1 = F'(z_0), \qquad c_2 = F''(z_0)/2\ .$$

In our case we calculate

$$\begin{aligned} F(z) &= \frac{1}{(z+1)^2} \\ F(1) &= \frac{1}{4} \\ F'(z) &= \frac{-2}{(z+1)^3} \\ F'(1) &= \frac{-2}{8} = -\frac{1}{4} \\ F''(z) &= \frac{6}{(z+1)^4} \\ F''(1) &= \frac{6}{16} = \frac{3}{8} \end{aligned}$$

so the series starts out

$$F(z) = \frac{1}{4} - \frac{(z-z_0)}{4} + \frac{3(z-z_0)^2}{16} + \cdots$$

and the Laurent series for f starts out

$$f(z) = \frac{1}{4(z-z_0)^2} - \frac{1}{4(z-z_0)} + \frac{3}{16} + \cdots$$

which makes the residue $-1/4$. So the integral is

$$-2\pi i/4 = -\pi i/2\ .$$

Exercise 3.1. *Write*

$$\frac{a_0 + a_1 x}{b_0 + b_1 x}$$

as a power series in x, assuming $b_0 \neq 0$. (Hint: Write first the series for $1/(b_0 + b_1 x)$, peeking at the remarks on geometric series later on in this chapter. Then multiply.

Exercise 3.2. *Write the first two terms of the power series of*

$$\frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2}$$

assuming $b_0 \neq 0$.

Exercise 3.3. *If*

$$f(z) = \frac{P(z)}{Q(z)}$$

with $P(z_0) \neq 0$ and $Q(z_0) = 0$ but $Q'(z_0) \neq 0$ then f has a simple pole and the residue is $P(z_0)/Q'(z_0)$, as we saw in the last chapter.

If $P(z_0) \neq 0$ and $Q(z_0) = Q'(z_0) = 0$ but $Q''(z_0) \neq 0$ f has a pole of order 2. Find a formula for the residue at z_0 in terms of

$$P(z_0), \quad P'(z_0), \quad Q''(z_0), \quad Q'''(z_0)\ .$$

(Hint: Write $Q(z) = (z - z_0)^2 R(z_0)$ and use the previous Exercise.)

Exercise 3.4. *Find*

$$\int_{-\infty}^{\infty} \frac{e^{-iax}}{(x^2+1)^2}\, dx$$

for all real numbers a.

Exercise 3.5. *Find*

$$\int_C \frac{1}{(z^2-1)^2}\, dz$$

where C is the circle of radius 2, oriented clockwise.

Exercise 3.6. *Find*

$$\int_C \frac{1}{(z^2-1)^3}\, dz$$

where C is the circle of radius 2, oriented clockwise.

Exercise 3.7. *Do the last two problems by expressing the function as a power series in $1/z$, and integrating term by term.*

4. Convergence

You learned in secondary school that the geometric series

$$1 + z + z^2 + z^3 + \cdots$$

converges to

$$\frac{1}{1-z}$$

if $|z| < 1$. The same is true even if z is allowed to be complex! We can then write

$$\frac{1}{z - z_0} = \frac{1}{z}\,\frac{1}{1 - (z_0/z)} = \frac{1}{z}\left(1 + (z_0/z) + (z_0/z)^2 + \cdots = \frac{1}{z} + \frac{z_0}{z^2} + \frac{z_0^2}{z^3} + \cdots\right.$$

To be continued . . .