Mathematics 307—October 16, 1995

Second homework solutions

Exercise 1. *If a particle with position vector* $\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ *is rotating clockwise around the axis* $x = y = z$ *(clockwise as seen looking from this vector towards the origin) with a speed of* 1^r *per second, what is its linear velocity?*

The vector $(1, -1, 2)$ lies on the same side of the plane $x + y + z = 0$ as $(1, 1, 1)$. Therefore a clockwise rotation is negative rotation around the axis $(1, 1, 1)$. So the calculation:

$$\begin{aligned}
\omega &= -(1/\sqrt{3})\,[\,1 \quad 1 \quad 1\,] \\
\mathbf{r} &= [\,1 \quad -1 \quad 2\,] \\
v &= \omega \times \mathbf{r} \\
&= [\,-1.73205 \quad 0.57735 \quad 1.15470\,]
\end{aligned}$$

Exercise 2. *Let* $u = (1, 1, 0)$, $v = (1, 2, 1)$. *What is the projection of* u *onto the line along* v*? The projection of* u *onto the plane perpendicular to* v*? The vector you get by rotating* u *by* $30°$ *around the axis along* v*?*

$$u_0 = \frac{u \bullet v}{v \bullet v}\, v = [\,0.5 \quad 1.0 \quad 0.5\,]\,, \qquad u_1 = u - \frac{u \bullet v}{v \bullet v}\, v = [\,0.5 \quad 0.0 \quad -0.5\,]$$

For the rotation: (1) It is simplest if we first normalize the axis to get

$$v = (1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6}) = (0.408248, 0.816497, 0.408248)\,.$$

(2) Express u as the sum of its two components u_0 and u_1. In the rotation, the parallel component u_0 remains fixed. The other gets rotated by $30°$ in the plane perpendicular to v. Let u_2 be what we get by rotating u_1 by $90°$ in this plane:

$$u_2 = v \times u_1 = (-0.408249, 0.408248, -0.408249)$$

Then u_1 is rotated to

$$(\cos 30°)u_1 + (\sin 30°)u_2 = (0.728889, 1.20412, -0.137137)\,.$$

You can also do this problem by calculating the matrix of rotation completely. Set

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix}, \qquad X = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

and the matrix of the rotation is is

$$XRX^{-1} = \begin{bmatrix} 0.888355 & -0.159466 & 0.430577 \\ 0.248782 & 0.955342 & -0.159466 \\ -0.385919 & 0.248782 & 0.888354 \end{bmatrix}$$

The vector we get here is

$$\begin{bmatrix} 0.888355 & -0.159466 & 0.430577 \\ 0.248782 & 0.955342 & -0.159466 \\ -0.385919 & 0.248782 & 0.888354 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.728889 \\ 1.204120 \\ -0.137137 \end{bmatrix}$$

In a program:

```
% u replaced by radius of u

/radius {
1 dict begin
/u exch def
u 0 get
dup mul
u 1 get
dup mul add
u 2 get
dup mul add
sqrt
end
} def

% u replaced by u/|u|

/normalize {
2 dict begin
/u exch def
/rad u radius def
[
u 0 get rad div
u 1 get rad div
u 2 get rad div
]
end
} def

% axis theta u replaced by rotation of u

/threerotation {
16 dict begin
/u exch def
/theta exch def
/axis exch def

/axis
axis
normalize def

/u0 axis u axis dotproduct vectorscale def
/u1 u u0 vectorsub def
/u2 axis u1 crossprod def

u0
u1 theta cos vectorscale
u2 theta sin vectorscale
vectoradd
vectoradd
% u0 + (u1 rotated theta degrees)
```

```
end
} def

axis
theta
u
threerotation
==
```

Exercise 3. *Suppose that*

$$T = I + \Omega\,\Delta t + \text{ terms of order } \Delta t^2$$

is an orthogonal transformation for all t. What condition must Ω satisfy? (Hint. Write out ${}^tT\,T$).

We must have ${}^tT\,T = I$. For $t = 0$ we have $T(t) = I$. Therefore for t near 0, $T(t)$ will be close to the identity matrix. This means that $T(t)$ for t near 0 will be I plus something which gets smaller as $\Delta t \to 0$. If

$$T(t) = \begin{bmatrix} t_{1,1}(t) & t_{1,2}(t) & t_{1,3}(t) \\ t_{2,1}(t) & t_{2,2}(t) & t_{2,3}(t) \\ t_{3,1}(t) & t_{3,2}(t) & t_{3,3}(t) \end{bmatrix}$$

then

$$\Omega = \begin{bmatrix} t'_{1,1}(0) & t'_{1,2}(0) & t'_{1,3}(0) \\ t'_{2,1}(0) & t'_{2,2}(0) & t'_{2,3}(0) \\ t'_{3,1}(0) & t'_{3,2}(0) & t'_{3,3}(0) \end{bmatrix}$$

The equation ${}^tT\,T = I$ must hold identically, and this means

$$I = (I + \Omega\,\Delta t + X\Delta t^2)(I + \Omega\,\Delta t + X\Delta t^2) = I + (\Omega + {}^t\Omega)\Delta t + (\ldots)\Delta t^2$$

The coefficient of Δt must vanish:

$$\Omega + {}^t\Omega = 0, \quad {}^t\Omega = -\Omega$$

or in other words Ω must be **skew-symmetric**. As an example let T be rotation by angle t in the plane. Then

$$\Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Exercise 4. *(a) Find the centre of gravity of a tennis racket. Assume it is constructed by adding a circle of radius 10 cm to a thin handle of length 20 cm, and that the linear density is 1 gm/cm around the circle, 2 gm/cm in the handle. This calculation will use the sum of two integrals, one over each component.*

(b) Find its moment of inertia matrix $\mathcal{I}$ with respect to its centre of gravity—its principal axes (clear) and eigenvalues.

First we calculate the moments of each of the two pieces, then add them together to get the total moments. At the start, pick the origin to be the point where the head and the handle are joined.

For the handle, y and z are both identically 0, so that all moments involving them vanish.

$$M_y = M_z = M_{xy} = M_{xz} = M_{yz} = M_{yy} = M_{zz} = 0$$

and

$$\begin{aligned}
M &= 40 \\
M_x &= \int x\,dm \\
&= 2\int_{-20}^{0} x\,dx \\
&= -400 \\
M_{xx} &= \int x^2\,dm \\
&= 2\int_{-20}^{0} x^2\,dx \\
&= 16000/3
\end{aligned}$$

For the circle, we can parametrize it by the angle θ at the centre. It is simplest to choose the origin at the centre, temporarily, and then shift to the circumference. Thus $x = 10\cos\theta$, $y = 10\sin\theta$, $dm = 10\,d\theta$. Again, since $z = 0$ identically, all moments involving it vanish.

$$M_z = M_{xz} = M_{yz} = M_{zz} = 0$$

The other moments *with respect to the centre*:

$$\begin{aligned}
M &= 20\pi \\
M_x &= \int_0^{2\pi} (10\cos\theta)10\,d\theta \\
&= 0 \\
M_y &= \int_0^{2\pi} (10\sin\theta)10\,d\theta \\
&= 0 \\
M_{xx} &= \int_0^{2\pi} (10\cos\theta)^2 10\,d\theta \\
&= 1000\int_0^{2\pi} \cos^2\theta\,d\theta \\
&= \frac{1000}{2}\int_0^{2\pi} (1+\cos 2\theta)\,d\theta \\
&= 1000\pi \\
M_{xy} &= \int_0^{2\pi} (10\cos\theta)(10\sin\theta)10\,d\theta \\
&= \frac{1000}{2}\int_0^{2\pi} \sin 2\theta\,d\theta \\
&= 0 \\
M_{yy} &= \int_0^{2\pi} (10\sin\theta)^2 10\,d\theta \\
&= 1000\int_0^{2\pi} \sin^2\theta\,d\theta \\
&= \frac{1000}{2}\int_0^{2\pi} (1-\cos 2\theta)\,d\theta \\
&= 1000\pi
\end{aligned}$$

After translation to the left 10 cm, only M_x and M_{xx} change to

$$\begin{aligned} M_x^{\#} &= M_x - (-10)M \\ &= 200\pi \\ M_{xx}^{\#} &= M_{xx} - (-10)M_x - (-10)M_x + 100M \\ &= M_{xx} + 100M \\ &= 3000\pi \end{aligned}$$

For the total moments we get

$$\begin{aligned} M^{tot} &= 40 + 20\pi \\ M_x^{tot} &= -400 + 200\pi \\ M_y^{tot} &= 0 \\ M_z^{tot} &= 0 \\ M_{xx}^{tot} &= 16000/3 + 3000\pi \\ M_{xy}^{tot} &= 0 \\ M_{yy}^{tot} &= 1000\pi \\ M_{xz}^{tot} &= 0 \\ M_{xz}^{tot} &= 0 \\ M_{zz}^{tot} &= 0 \end{aligned}$$

Therefore the coordinates of the centre of gravity are

$$\begin{aligned} t_x &= \frac{-400 + 200\pi}{40 + 20\pi} \\ &= 2.2203 \\ t_y &= 0 \\ t_z &= 0 \end{aligned}$$

After we shift coordinates again to this, we get quadratic moments

$$\begin{aligned} M_{xx}^* &= M_{xx}^{tot} - t_x M_x^{tot} - t_x M_x^{tot} + M t_x^2 \\ &= M_{xx}^{tot} - 2\frac{M_x^{tot} M_x^{tot}}{M} + M \frac{M_x}{M}\frac{M_x}{M} \\ &= M_{xx}^{tot} - \frac{M_x^{tot} M_x^{tot}}{M} \\ &= (16000/3 + 3000\pi) + \frac{(-400 + 200\pi)^2}{40 + 20\pi} \\ &= 15265.3 \\ M_{yy}^* &= 1000\pi \\ &= 3141.6 \\ M_{zz}^* &= 0 \end{aligned}$$

and all others vanish.

$$\mathcal{I} = \begin{bmatrix} M_{yy}^* & 0 & 0 \\ 0 & M_{xx}^* & 0 \\ 0 & 0 & M_{xx}^* + M_{yy}^* \end{bmatrix}$$

Note that it is already diagonal, so the principal axes are the coordinate axes.

Exercise 5. *Do the same for a system made up of three objects: (i) mass 3, location* $-\mathbf{i}$*; (ii) mass 1, location* $\mathbf{i}+\mathbf{j}$*; (iii) mass 2, location* $\mathbf{i}-\mathbf{j}$*.*

$$\begin{aligned} M &= 6 \\ M_x &= -3+1+2 \\ &= 0 \\ M_y &= 1-2 \\ &= -1 \\ M_z &= 0 \\ M_{xx} &= 3+1+2 \\ &= 6 \\ M_{xy} &= 1-2 \\ &= -1 \\ M_{yy} &= 1+2 \\ &= 3 \end{aligned}$$

The centre of gravity is therefore

$$(t_x, t_y, t_z) = (0, -1/6, 0)$$

and the moments with respect to it

$$\begin{aligned} M^*_{xx} &= M_{xx} - 2t_x M_x + M t_x^2 \\ &= M_{xx} - \frac{M_x^2}{M} \\ &= 6 \\ M^*_{xy} &= M_{xy} - t_y M_x - t_x M_y + M t_x t_y \\ &= M_{xy} - \frac{M_x M_y}{M} \\ &= -1 \\ M^*_{yy} &= M_{yy} - t_y M_x - t_x M_y + M t_y^2 \\ &= M_{yy} - \frac{M_y^2}{M} \\ &= 3 - \frac{1}{6} \end{aligned}$$

all others vanishing, since z vanishes. In this case the matrix $\mathcal{I}$ is

$$\mathcal{I} = \begin{bmatrix} M^*_{yy} & -M^*_{xy} & 0 \\ -M^*_{xy} & M^*_{xx} & 0 \\ 0 & 0 & M^*_{xx} + M^*_{yy} \end{bmatrix} = \begin{bmatrix} 2.83333 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 8.83333 \end{bmatrix}$$

It is not difficult to find the eigenvalues and eigenvectors of this because it comes down to a 2×2 problem.

Eigenvalues: 8.33333, 6.28935, 2.54938.

Eigenvectors: $(0, 0, 1)$, $(0.277949, 0.960596, 0)$, $(-0.960596, 0.277949)$.

Exercise 6. *How does differentiation act on the space of functions of the form*

$$a \cos \omega x + b \sin \omega x?$$

Choose a basis and write down the matrix.

Basis $\cos\omega x$, $\sin\omega x$. Matrix

$$\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

Exercise 7. *How does differentiation act on the space of polynomials of degree at most n? Choose a basis and write down the matrix.*

Basis $e_m = x^m/m!$ The transformation T takes e_m to e_{m-1}. Matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Exercise 8. *Find a formula for*

$$\int x^n e^{cx}\,dx$$

by this method.

We have

$$x^n e^{cx} = n!\frac{x^n e^{cx}}{n!}$$

and its integral is $n!$ times

$$\frac{x^n}{n!}\frac{e^{cx}}{c} - \frac{x^{n-1}}{(n-1)!}\frac{e^{cx}}{c^2} + \cdots \pm \frac{e^{cx}}{c^{n+1}}$$

Exercise 9. *Let T be the linear operator*

$$Tf = f'' + f$$

acing on the space of functions $P(x)e^{-x}$ where $P(x)$ has degree at most 4. What is its matrix?

Choose basis $x^n e^{-x}/n!$. Then $f \mapsto f'$ has matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Since $x^2+1 = (x+i)(x-i)$ the operator $f''+f$ has matrix on this space

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Exercise 10. *There exists a unique solution of the form $P(x)e^{-x}$ of the differential equation*

$$y'' + y = x^4 e^{-x}$$

Find it by this method, considering the operator $y \mapsto y'' + y$ as a linear operator.

Apply the inverse of the matrix in the previous exercise to the function $x^4 e^{-x}$, whose coordinates are $(0,0,0,0,24)$.

$$\begin{bmatrix} 2 & -2 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{bmatrix}$$

We can write the inverse as

$$\frac{1}{2}\left(I + \frac{N}{2}\right)^{-1} = \left(\frac{I}{2} - \frac{N}{4} + \frac{N^2}{8} - \frac{N^3}{16} + \frac{N^4}{32}\right)$$

where

$$\frac{N}{2} = \begin{bmatrix} 0 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{N^2}{4} = \begin{bmatrix} 0 & 0 & 1/4 & -1/4 & 1/8 \\ 0 & 0 & 0 & 1/4 & -1/4 \\ 0 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{N^3}{8} = \begin{bmatrix} 0 & 0 & 0 & -1/8 & 3/8 \\ 0 & 0 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{N^4}{16} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the solution is

$$[-3.0 \quad 0.0 \quad 6.0 \quad 12.0 \quad 12.0]$$

Exercise 11. *How does $T : f \mapsto f' + f$ act on the space $P(x)e^{-x}$ with P of degree at most 3? Choose a basis and write down the matrix.*

The transformation T takes Pe^{-x} to $P'e^{-x}$. Basis e^{-x}, xe^{-x}, $x^2e^{-x}/2$, $x^3e^{-x}/6$. Matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 12. *Suppose that $u = (a, b, c)$ is a 3D vector. The map taking v to the cross-product $u \times v$ is a 3D linear transformation. What is its matrix? What are its eigenvectors and values?*

The matrix has as columns $\omega \times \mathbf{i}$, etc. It is

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

Geometrically, the linear transformation has this effect: it takes u to 0, since $u \times u = 0$, and rotates vectors perpendicular to u by $90°$, then scales them.

Therefore u is an eigenvector with eigenvalue 0. On the plane perpendicular to u it is rotation by $90°$ followed by scaling by the factor $||u|| = \sqrt{a^2 + b^2 + c^2}$. Its eigenvalues are therefore $\pm\sqrt{a^2 + b^2 + c^2}$, eigenvectors $u_1 \mp i u_1$ if u_1, u_2 are a pair of orthogonal unit vectors in this plane.