Mathematics 307—November 9, 1995

Third homework solutions

Exercise 1. *Let*

$$M = \begin{bmatrix} 1 & 3 & -1 \\ -1 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

Apply the Gram-Schmidt process to write $M = KAN$ where K is an orthogonal matrix, A a diagonal one, N an upper triangular one with 1's along the diagonal.

We apply the formulas and multiply M on the right by the matrices

$$\begin{bmatrix} 1 & 1/6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 15/83 \\ 0 & 0 & 1 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 3.16667 & 0.0722891 \\ -1 & 1.83333 & -0.168675 \\ -2 & 0.666667 & 0.120482 \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 2.44949 & 0 & 0 \\ 0 & 3.71932 & 0 \\ 0 & 0 & 0.219528 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.408248 & 0.85141 & 0.329293 \\ -0.408248 & 0.492922 & -0.76835 \\ -0.816497 & 0.179244 & 0.548821 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & -0.166667 & -0.5 \\ 0 & 1 & -0.180723 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 2. *Make a vector space from the polynomials of degree at most 5. Put a dot product on this vector space according to the formula*

$$P \bullet Q = \int_{-1}^{1} P(x)Q(x)\,dx$$

Start with the basis 1, x, ..., x^5 and apply the Gram-Schmidt process to it to find an orthogonal basis. Then find an orthonormal one.

The point is to think of polynomials as vectors but assigned a new length formula. This idea arises from physics in spherical coordinates, where we are interested in finding the energy of various modes of motion of a sphere which are rotationally invariant. A polynomial $P(x)$ corresponds to a function $P(\cos\varphi)$, where φ is latitude, and energy is represented in spherical coordinates as

$$2\pi \int_{-\pi}^{\pi} P(\cos\varphi)\,\sin\varphi\,d\varphi = 2\pi \int_{-1}^{1} P(x)\,dx$$

The first new function is of course just 1. The second is x itself, since x and 1 are orthogogoanl, which means in this context that

$$\int_{-1}^{1} x\,dx = 0$$

Similarly all x^i and x^j are orthogonal if i and j have different parity. But the next example gives the new third vector as

$$P_2(x) = x^2 \frac{x^2 \bullet 1}{1 \bullet 1} 1 = x^2 - \frac{2/3}{2} = x^2 - 1/3$$

and the new fourth one is

$$x^4 - \frac{x^4 \bullet P_2}{P_2 \bullet P_2} P_2 - \frac{x^4 \bullet P_0}{P_0 \bullet P_0} P_0$$

The total list is

$$\begin{aligned} P_0(x) &= 1 \\ P_1 &= x \\ P_2 &= x^2 - 1/3 \\ P_3 &= x^3 - 3/5x \\ P_4 &= x^4 - 6/7\,x^2 + 3/35 \\ P_5 &= x^5 - 10/9\,x^3 + 5/21\,x \end{aligned}$$

The next stage is to normalize them. The list of squared lengths with respect to this dot product is

$$\begin{aligned} ||P_0||^2 &= 2 \\ ||P_1||^2 &= 2/3 \\ ||P_1||^2 &= 0.177778 \\ ||P_3||^2 &= 0.0457143 \\ ||P_4||^2 &= 0.01161 \\ ||P_5||^2 &= 0.00293183 \end{aligned}$$

Exercise 3. *Find the WLU factorization of the matrices*

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Of the similar 10×10 matrix? $N \times N$?

Find the inverse of the 10×10 matrix in this series

In every case $W = 1$—there is no necessity to pivot. The successive LU factorizations give for example

$$U_2 = \begin{bmatrix} 2 & -1 \\ 0.0 & 1.5 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}$$

$$U_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0.0 & 1.5 & -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.33333 & -1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.25 & -1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 \\ 0.0 & -0.666667 & 1 & 0 & 0 \\ 0.0 & 0.0 & -0.75 & 1 & 0 \\ 0.0 & 0.0 & 0.0 & -0.8 & 1 \end{bmatrix}$$

From which you can guess the rule.

As for inverse, we write $M = LU$, $M^{-1} = U^{-1}L^{-1}$. We have

$L^{-1} =$

$$\begin{bmatrix}
1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.5 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.333333 & 0.666667 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.25 & 0.5 & 0.75 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.2 & 0.4 & 0.6 & 0.8 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.166667 & 0.333333 & 0.5 & 0.666667 & 0.833333 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.142857 & 0.285714 & 0.428571 & 0.571429 & 0.714286 & 0.857143 & 1.0 & 0.0 & 0.0 & 0.0 \\
0.125 & 0.25 & 0.375 & 0.5 & 0.625 & 0.75 & 0.875 & 1.0 & 0.0 & 0.0 \\
0.111111 & 0.222222 & 0.333333 & 0.444444 & 0.555555 & 0.666667 & 0.777778 & 0.888889 & 1.0 & 0.0 \\
0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1.0
\end{bmatrix}$$

$U^{-1} =$

$$\begin{bmatrix}
0.5 & 0.333333 & 0.25 & 0.2 & 0.166667 & 0.142857 & 0.125 & 0.111111 & 0.1 & 0.0909091 \\
0.0 & 0.666667 & 0.5 & 0.4 & 0.333333 & 0.285714 & 0.25 & 0.222222 & 0.2 & 0.181818 \\
0.0 & 0.0 & 0.75 & 0.6 & 0.5 & 0.428571 & 0.375 & 0.333333 & 0.3 & 0.272727 \\
0.0 & 0.0 & 0.0 & 0.8 & 0.666667 & 0.571429 & 0.5 & 0.444444 & 0.4 & 0.363636 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.833333 & 0.714286 & 0.625 & 0.555555 & 0.5 & 0.454545 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.857143 & 0.75 & 0.666667 & 0.6 & 0.545454 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.875 & 0.777778 & 0.7 & 0.636364 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.888889 & 0.8 & 0.727273 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.818182 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.909091
\end{bmatrix}$$

$M^{-1} =$

$$\begin{bmatrix}
0.909091 & 0.818182 & 0.727273 & 0.636364 & 0.545455 & 0.454545 & 0.363636 & 0.272727 & 0.181818 & 0.0909091 \\
0.818182 & 1.63636 & 1.45455 & 1.27273 & 1.09091 & 0.909091 & 0.727273 & 0.545454 & 0.363636 & 0.181818 \\
0.727273 & 1.45455 & 2.18182 & 1.90909 & 1.63636 & 1.36364 & 1.09091 & 0.818182 & 0.545454 & 0.272727 \\
0.636364 & 1.27273 & 1.90909 & 2.54545 & 2.18182 & 1.81818 & 1.45455 & 1.09091 & 0.727273 & 0.363636 \\
0.545455 & 1.09091 & 1.63636 & 2.18182 & 2.72727 & 2.27273 & 1.81818 & 1.36364 & 0.909091 & 0.454545 \\
0.454545 & 0.909091 & 1.36364 & 1.81818 & 2.27273 & 2.72727 & 2.18182 & 1.63636 & 1.09091 & 0.545454 \\
0.363636 & 0.727273 & 1.09091 & 1.45455 & 1.81818 & 2.18182 & 2.54545 & 1.90909 & 1.27273 & 0.636364 \\
0.272727 & 0.545455 & 0.818182 & 1.09091 & 1.36364 & 1.63636 & 1.90909 & 2.18182 & 1.45455 & 0.727273 \\
0.181818 & 0.363636 & 0.545455 & 0.727273 & 0.909091 & 1.09091 & 1.27273 & 1.45455 & 1.63636 & 0.818182 \\
0.0909091 & 0.181818 & 0.272727 & 0.363636 & 0.454545 & 0.545455 & 0.636364 & 0.727273 & 0.818182 & 0.909091
\end{bmatrix}$$

Exercise 4. *Find the WLU factorization of the matrices*

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Of the similar 10×10 matrix? $N \times N$?

Find the inverse of the 10×10 matrix in this series

Again, no swaps necessary. Here are a few of the U:

$$U = \begin{bmatrix} 4 & 1 \\ 0.0 & 3.75 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 0.25 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 1 & 0 \\ 0.0 & 3.75 & 1.0 \\ 0.0 & 0.0 & 3.73333 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0.0 & 0.266667 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0.0 & 3.75 & 1.0 & 0.0 \\ 0.0 & 0.0 & 3.73333 & 1.0 \\ 0.0 & 0.0 & 0.0 & 3.73214 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0.0 & 0.266667 & 1 & 0 \\ 0.0 & 0.0 & 0.267857 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0 & 3.75 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 3.73333 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.73214 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 3.73206 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 3.73205 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 3.73205 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 3.73205 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 3.73205 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0 & 0.266667 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0 & 0.0 & 0.267857 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.0 & 0.0 & 0.0 & 0.267943 & 1 & 0 & 0 & 0 & 0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.267949 & 1 & 0 & 0 & 0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.267949 & 1 & 0 & 0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.267949 & 1 & 0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.267949 & 1 \end{bmatrix}$$

An interesting pattern.

Exercise 5. *Find the WLU factorization of the matrix*

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Do this by hand, and then by computer. How different are the answers?

Here is what happens for the first few matrices of this type:

$$\begin{bmatrix} 1.0 & 0.5 & 0.333333 \\ 0.0 & 0.0833333 & 0.0833333 \\ 0.0 & 0.0 & 0.00555557 \end{bmatrix}$$

$$\begin{bmatrix} 1.0 & 0.5 & 0.333333 & 0.25 \\ 0.0 & 0.0833333 & 0.0833333 & 0.075 \\ 0.0 & 0.0 & 0.00833335 & 0.0128572 \\ 0.0 & 0.0 & 0.0 & -0.000238096 \end{bmatrix}$$

$$\begin{bmatrix} 1.0 & 0.5 & 0.333333 & 0.25 & 0.2 \\ 0.0 & 0.0833333 & 0.0833333 & 0.075 & 0.0666667 \\ 0.0 & 0.0 & 0.00952382 & 0.015 & 0.0177778 \\ 0.0 & 0.0 & 0.0 & -0.000416664 & -0.000846558 \\ 0.0 & 0.0 & 0.0 & 0.0 & -1.13457e-05 \end{bmatrix}$$

As we shall see next week, the last cannot possibly be correct. Even the first is slightly–very slightly—suspicious.

Exercise 6. *Find the WLU factorization of*

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ -6 & -5 & -8 \end{bmatrix}$$

showing essentially all the intermediate steps.

$$U = \begin{bmatrix} -6 & -5 & -8 \\ 0.0 & 1.33333 & -1.66667 \\ 0.0 & 0.0 & -0.25 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -0.333333 & 1 & 0 \\ -0.333333 & 0.25 & 1 \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$