Mathematics 307—December 5, 1995

Fourth homework solutions

Exercise 1. *Find the solutions of*

$$\begin{aligned} 0.999x + y &= 1.000 \\ x + 0.999y &= 0.999 \end{aligned}$$

and then

$$\begin{aligned} 0.999x + y &= 0.999 \\ x + 0.999y &= 1.000 \end{aligned}$$

and explain carefully why the answers are so different.

The solutions are $(0,1)$ and $(1,0)$. So that even though the right hand sides of the systems are very close, the solutions are very far apart. Why is that? When can it happen in general?

The eigenvectors of the matrix

$$\begin{bmatrix} 0.999 & 1.000 \\ 1.000 & 0.999 \end{bmatrix}$$

are $(1,1)$ and $(1,-1)$, with eigenvalues 1.999 and -0.001. Note that A^{-1} therefore has eigenvalues $1/1.999$ and -1000. Solving the systems amounts to calculating

$$\begin{bmatrix} 0.999 & 1.000 \\ 1.000 & 0.999 \end{bmatrix}^{-1} \begin{bmatrix} 1.000 \\ 0.999 \end{bmatrix}, \quad \begin{bmatrix} 0.999 & 1.000 \\ 1.000 & 0.999 \end{bmatrix}^{-1} \begin{bmatrix} 0.999 \\ 1.000 \end{bmatrix}$$

Let $u_1 = (0.999, 1.000)$ and $u_2 = (1.000, 0.999)$. Then

$$\delta = u_2 - u_1 = (0.001, -0.001).$$

which is an eigenvector for the inverse matrix with eigenvalue -1000. Therefore

$$A^{-1}u_2 = A^{-1}u_1 + A^{-1}\delta$$

and the two solutions will differ by $A^{-1}\delta = (1,-1)$, which is relatively large.

This sort of thing can happen whenever the rows of A are somewhat more linearly dependent than average. In these circumstances, A is said to be **badly conditioned**, and in general a small variation in c will lead to a relatively large variation in the solution to $Ax = c$. This reflects a real difficulty, not a poor technique for solving the system. It means that in some situations rounding errors can have serious effects. This is just a fact of life.

Exercise 2. *Find the singular value decomposition of the matrix*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$$

We must first calculate

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 20 & 32 \\ 20 & 29 & 47 \\ 32 & 47 & 77 \end{bmatrix}$$

We must find the eigenvalues and eigenvectors of M. The most serious thing to decide is how to do it, if you don't have a calculator which will tell you the answer. The simplest thing to do is probably to find the roots of the characteristic equation.

In fact, as soon as we calculate the determinant we see that this matrix is singular, since its determinant is 0. Therefore one of its eigenvalues is 0. The eigenvector equations are

$$\begin{aligned} 14x + 20y + 32z &= 0 \\ 20x + 29y + 47z &= 0 \end{aligned}$$

witha solution $(-2, 3, 1)$. A normalized eigenvector for 0 is

$$v_1 = (-2, 3, -1)/\sqrt{4+9+1} = (-0.534522, 0.801784, -0.267261)$$

You must now find the second and third eigenvectors and values. There are many ways to do this. The simplest is to **deflate**, looking at the two-dimensional transformation you get by restricting M to the plane perpendicular to v_1. We must first find a basis perpendicular to v_1. We solve the equation

$$2x - 3y + z = 0$$

to get

$$v_2 = (-1, 0, 2)/\sqrt{5} = (-0.447214, 0.000000, 0.894427)$$

and taking cross product

$$v_3 = (0.717137, 0.597614, 0.358569)$$

If we set

$$V = [v1\ v2\ v3] = \begin{bmatrix} -0.534522 & -0.447214 & 0.717137 \\ 0.801784 & 0.000000 & 0.597614 \\ -0.267261 & 0.894427 & 0.358569 \end{bmatrix}$$

then

$$V^{-1}MV = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 38.8000 & 55.3765 \\ 0.0000 & 55.3765 & 81.2000 \end{bmatrix}$$

The eigenvalues of this 2×2 matrix are 119.296 and $84/119.296 = 0.704131$. Eigenvectors are

$$(0.56678, 0.82387), \quad (-0.82387, 0.56678)$$

We have therefore

$$\begin{bmatrix} 1.00000 & 0.00000 & 0.00000 \\ 0.00000 & 0.566777 & 0.823872 \\ 0.00000 & -0.823872 & 0.566777 \end{bmatrix} \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 38.8000 & 55.3765 \\ 0.0000 & 55.3765 & 81.2000 \end{bmatrix} \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.566777 & -0.823872 \\ 0.000000 & 0.823872 & 0.566777 \end{bmatrix}$$
$$= \begin{bmatrix} 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 119.296 & 0.00000 \\ 0.00000 & 0.00000 & 0.704131 \end{bmatrix}$$

Let

$$U = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.566777 & -0.823872 \\ 0.000000 & 0.823872 & 0.566777 \end{bmatrix}$$
$$M_* = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 38.8000 & 55.3765 \\ 0.0000 & 55.3765 & 81.2000 \end{bmatrix}$$
$$D = \begin{bmatrix} 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 119.296 & 0.00000 \\ 0.00000 & 0.00000 & 0.704131 \end{bmatrix}$$

So

$$V^{-1}MV = M_*, \quad U^{-1}M_*U = D$$

Therefore also

$$U^{-1}V^{-1}MVU = D$$

so the eigenvector matrix for the original matrix M is

$$\begin{aligned} X &= VU \\ &= \begin{bmatrix} -0.534522 & -0.447214 & 0.717137 \\ 0.801784 & 0.000000 & 0.597614 \\ -0.267261 & 0.894427 & 0.358569 \end{bmatrix} \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.566777 & -0.823872 \\ 0.000000 & 0.823872 & 0.566777 \end{bmatrix} \\ &= \begin{bmatrix} 0.534522 & 0.337358 & 0.774904 \\ 0.801784 & 0.492357 & 0.338714 \\ -0.267261 & 0.802356 & -0.533665 \end{bmatrix} \end{aligned}$$

We now want to find an orthogonal matrix Y such that

$$A = XDY$$

where

$$D = \begin{bmatrix} 0.000000 & 0.00000 & 0.000000 \\ 0.000000 & 10.9223 & 0.000000 \\ 0.000000 & 0.00000 & 0.839125 \end{bmatrix}$$

This is a peculiar case, since D is singular. The method I explained in class notes doesn't work, but a simple modification does. Then

$$X^{-1}A = \begin{bmatrix} 0.000000 & 0.00000 & 0.00000 \\ 4.5315 & 6.16357 & 7.79564 \\ -0.682328 & -0.102374 & 0.477579 \end{bmatrix}$$

which we want to write as DY. If

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(y_i a row of Y) then

$$DY = \begin{bmatrix} d_1 y_1 \\ d_2 y_2 \\ d_3 y_3 \end{bmatrix}$$

Since Y is an orthogonal matrix, its rows have unit length. Therefore we can take the second and third rows of Y to be the normalized rows of $X^{-1}A$ and the first row of Y to be anything orthogonal to the first two, say their cross-product.

$$Y = \begin{bmatrix} 0.408248 & -0.816497 & 0.408248 \\ 0.414886 & 0.564312 & 0.713738 \\ -0.813141 & 0.122001 & 0.569138 \end{bmatrix}$$

Exercise 3. *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

by Jacobi's method, showing all intermediate steps.

```
[[1.26759 0.00000 0.411856 0.31479]
 [0.00000 0.0657414 0.0631327 0.0583704]
 [0.411856 0.0631327 0.2 0.166667]
 [0.31479 0.0583704 0.166667 0.142857]]

[[1.40801 0.0203728 0.00000 0.351733]
 [0.0203728 0.0657414 0.0597552 0.0583704]
 [0.00000 0.0597552 0.0595827 0.0561681]
 [0.351733 0.0583704 0.0561681 0.142857]]

[[1.49922 0.0343725 0.0140992 0.00000]
 [0.0343725 0.0657414 0.0597552 0.0513876]
 [0.0140992 0.0597552 0.0595827 0.0543697]
 [0.00000 0.0513876 0.0543697 0.0516459]]

[[1.49922 0.0346323 -0.0134484 0.00000]
 [0.0346323 0.122497 0.00000 0.0747027]
 [-0.0134484 0.00000 0.00282758 0.00403289]
 [0.00000 0.0747027 0.00403289 0.0516459]]

[[1.49922 0.0292687 -0.0134484 -0.0185132]
 [0.0292687 0.169748 0.00215584 0.00000]
 [-0.0134484 0.00215584 0.00282758 0.0034083]
 [-0.0185132 0.00000 0.0034083 0.00439447]]

[[1.49986 0.00000 -0.0133977 -0.0185088]
 [0.00000 0.169104 0.00245118 0.000407278]
 [-0.0133977 0.00245118 0.00282758 0.0034083]
 [-0.0185088 0.000407278 0.0034083 0.00439447]]

[[1.50009 -5.03957e-06 -0.0134389 0.00000]
 [-5.03957e-06 0.169104 0.00245118 0.000407247]
 [-0.0134389 0.00245118 0.00282758 0.00324226]
 [0.00000 0.000407247 0.00324226 0.00416553]]

...

[[1.50021 2.77172e-12 -5.24846e-08 0]
 [2.77172e-12 0.169141 -1.85979e-07 1.5584e-10]
 [-5.24846e-08 -1.85979e-07 0.00673836 -6.42965e-13]
 [0 1.5584e-10 -6.42965e-13 9.67979e-05]]

[1.50021 0.169141 0.00673836 9.67979e-05]
```

Exercise 4. *Find the eigenvalues and eigenvalues of the matrix*

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

by Jacobi's method.

```
[[3.73205 -2.45762e-07 1.30655e-11 1.94101e-15 9.69227e-11]
 [-2.45762e-07 3.0 1.28777e-06 1.29499e-12 -6.43735e-09]
 [1.30655e-11 1.28777e-06 2.0 2.25506e-06 -6.21933e-08]
 [1.94101e-15 1.29499e-12 2.25506e-06 1.0 0]
 [9.69227e-11 -6.43735e-09 -6.21933e-08 0 0.267949]]
```

Exercise 5. *Find the highest eigenvalue of the 5×5 Hilbert matrix by the power method, correct to 8 decimals. How many iterations would it take to find it correctly to 12 decimals?*

```
u = 1.000000000000, 0.000000000000, 0.000000000000, 0.000000000000, 0.000000000000,
eigenvalue approximation =  1.  000000000000
u = 0.826584298074, 0.413292149037, 0.275528099358, 0.206646074518, 0.165316859615,
eigenvalue approximation =  1.4 63611111111
u = 0.776137568107, 0.441907896056, 0.315648497183, 0.247257989460, 0.203856321097,
eigenvalue approximation =  1.5 52422525283
u = 0.768965206930, 0.445280563090, 0.320792913656, 0.252618579926, 0.209029878940,
eigenvalue approximation =  1.56 5086520973
u = 0.768002658837, 0.445723223795, 0.321473845202, 0.253329819454, 0.209717174343,
eigenvalue approximation = 1.56 6788988321
u = 0.767874422578, 0.445782034769, 0.321564396095, 0.253424422522, 0.209808602925,
eigenvalue approximation = 1.5670 15859403
u = 0.767857355016, 0.445789859403, 0.321576444982, 0.253437010925, 0.209820769059,
eigenvalue approximation = 1.5670 46055791
u = 0.767855083715, 0.445790900633, 0.321578048360, 0.253438686103, 0.209822388047,
eigenvalue approximation = 1.567050 074256
u = 0.767854781462, 0.445791039193, 0.321578261728, 0.253438909026, 0.209822603492,
eigenvalue approximation = 1.5670506 09012
u = 0.767854741240, 0.445791057632, 0.321578290122, 0.253438938692, 0.209822632163,
eigenvalue approximation = 1.5670506 80175
u = 0.767854735887, 0.445791060086, 0.321578293900, 0.253438942639, 0.209822635978,
eigenvalue approximation = 1.5670506 89645
u = 0.767854735175, 0.445791060413, 0.321578294403, 0.253438943165, 0.209822636486,
eigenvalue approximation = 1.56705069 0905
u = 0.767854735080, 0.445791060456, 0.321578294470, 0.253438943234, 0.209822636553,
eigenvalue approximation = 1.5670506910 72
u = 0.767854735068, 0.445791060462, 0.321578294479, 0.253438943244, 0.209822636562,
eigenvalue approximation = 1.56705069109 5
u = 0.767854735066, 0.445791060463, 0.321578294480, 0.253438943245, 0.209822636563,
eigenvalue approximation = 1.567050691098
u = 0.767854735066, 0.445791060463, 0.321578294480, 0.253438943245, 0.209822636564,
eigenvalue approximation = 1.567050691098}
```

Exercise 6. *Draw the curves*

$$x^2 + 2xy + 3y^2 = 1, \quad x^2 - 2xy + 3y^2 = 1$$

Completing the square, we can see they are both ellipses. Their matrices

$$\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 3 \end{bmatrix}$$

even have the same characteristic polynomial

$$\lambda^2 - 4\lambda + 2 = 0$$

with roots

$$2 \pm \sqrt{2}$$

The directions differ, since the eigenvectors are

$$\begin{bmatrix} \pm 1 \\ (\lambda - 1) \end{bmatrix} = \begin{bmatrix} \pm 1 \\ (1 \pm \sqrt{2}) \end{bmatrix}$$

...

Exercise 7. *Write down the full expression for the determinant of*

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$+\ 1\ 2\ 3\ 4$	$a_{1,1}a_{2,2}a_{3,3}a_{4,4}$
$-\ 1\ 2\ 4\ 3$	
$-\ 1\ 3\ 2\ 4$	
$+\ 1\ 3\ 4\ 2$	
$+\ 1\ 4\ 2\ 3$	
$-\ 1\ 4\ 3\ 2$	$-a_{1,1}a_{2,4}a_{3,3}a_{4,2}$
$-\ 2\ 1\ 3\ 4$	
$+\ 2\ 1\ 4\ 3$	
$+\ 2\ 3\ 1\ 4$	
$-\ 2\ 3\ 4\ 1$	
$-\ 2\ 4\ 1\ 3$	
$+\ 2\ 4\ 3\ 1$	
$+\ 3\ 1\ 2\ 4$	
$-\ 3\ 1\ 4\ 2$	
$-\ 3\ 2\ 1\ 4$	$-a_{1,3}a_{2,2}a_{3,1}a_{4,4}$
$+\ 3\ 2\ 4\ 1$	
$+\ 3\ 4\ 1\ 2$	
$-\ 3\ 4\ 2\ 1$	
$+\ 4\ 1\ 3\ 2$	
$-\ 4\ 1\ 2\ 3$	
$-\ 4\ 3\ 1\ 2$	
$+\ 4\ 3\ 2\ 1$	
$+\ 4\ 2\ 1\ 3$	
$-\ 4\ 2\ 3\ 1$	

Exercise 8. *If you apply Gaussian elimination to a tridiagonal $n \times n$ matrix, and you don't have to do any swaps, how many multiplications can you expect to perform? If you apply Gaussian elimination to an arbitrary $n \times n$ matrix?*

(1) To go from $n \times n$ to $(n-1) \times (n-1)$ takes $a_{2,1}$ to 0, $a_{2,2}$ to $a_{2,2} - (a_{2,1}/a_{1,1})a_{1,2}$. This involves one division and one multiplication. We do this $n-1$ times, so answer is $2(n-1)$.

(2) to go from $n \times n$ to $(n-1) \times (n-1)$ takes for each row below the first takes one division to get $p = a_{r,1}/a_{1,1}$ and then $n-1$ multiplications to replace $a_{r,i}$ bu $a_{r,i} - pa_{1,i}$. So the final answer is

$$[(n-1) + (n-1)(n-1)] + [(n-2) + (n-2)(n-2)] + \cdots + 2.1$$

There is an exact expression for this, but it is messy. A good approximation sees this as close to

$$\int_1^n x(x-1)\,dx \sim n^3/3$$

Exercise 9. *Find the generalized eigenvalues and eigenvectors of the problem*

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} v = \lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} v$$

Recall, we want to solve

$$Kv = \lambda M v$$

(1) We apply Gauss elimination to set $M = L\,U_*$, then in turn equal to $L\,D\,{}^tL$ and $M = {}^tU$ where $U = \sqrt{D}L$. We then have

$${}^tU^{-1} M U^{-1}\, U v = \lambda U v$$

so

$$K_* v_* = \lambda v_*$$

so we must solve this and then set $v = U^{-1} v_*$.

The eigenvalues we are looking for are those of the matrix

$$K_* = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}$$

and if v_* is an eigenvector for this matrix then

$$\begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix} v_*$$

is one for the original problem.

Exercise 10. *Explain why the matrix*

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \end{bmatrix}$$

is positive definite.

The corresponding quadratic function is

$$4x_1^2 + 2x_1x_2 + 4x_2^2 + \cdots + 4x_6^2 = 3x_1^2 + (x_1 + x_2)^2 + 2x_2^2 + \cdots + 3x_6^2$$