Mathematics 307—October 25, 1995

The determinant

There are several ways to calculate the determinant, and it is not obvious that they all agree. For this reason it is important to have a single definition of the determinant from one which one can deduce that the different ways to calculate it are valid.

Let

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & & & \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix}$$

be an $n \times n$ matrix. (The determinant is defined only for square matrices.) Then the determinant is defined to be the sum of a certain number of terms, each with a $\pm$ attached to it. Each one of these terms is the product of n coefficients from the matrix. How do we decide which coefficients and how do we assign the sign?

To get the terms, we choose one entry from each row, with the restriction that no two items come from the same column.

For example, let's look at a 4×4 matrix. One way we might make these choices is by taking the first item from the first row, the second from the second row, etc.

$$\begin{bmatrix} \mathbf{a}_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & \mathbf{a}_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & \mathbf{a}_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & \mathbf{a}_{4,4} \end{bmatrix}$$

which gives $a_{1,1}a_{2,2}a_{3,3}a_{4,4}$. But we might also choose, say, in this fashion

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \mathbf{a}_{1,3} & a_{1,4} \\ \mathbf{a}_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & \mathbf{a}_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & \mathbf{a}_{4,4} \end{bmatrix}$$

which gives $a_{1,3}a_{2,1}a_{3,2}a_{4,4}$. How many different terms do we get in this way? We have n different choices possible from the first row, but having made that we have $n-1$ choices from the second, then $n-2$ from the third etc. All these choices are independant of each other, so we get

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$$

all in all. That's a lot.

Any set of choices is determined by the sequence of columns we choose the items from. Thus in the first example we get the sequence 1 2 3 4 and in the second 3 1 2 4. If $i_1\, i_2\, i_3\, \dots i_n$ is this list then the corresponding product is

$$a_{1,i_1} a_{2,i_2} \dots a_{n,i_n} .$$

The only thing left to be decided is the sign to be attached to this in making up the determinant.

An **inversion** in a sequence $i_1\, i_2\, i_3 \dots i_n$ is a pair which are in the wrong order. The sign of the term corresponding to the sequence is $+$ if the number of inversions is even, otherwise $-$.

For example, in the sequence

1 2 3 4

there are no inversions, so the sign is +. In the sequence

$$3\,1\,2\,4$$

the inversions are

$$(3,1)\,(3,2)$$

so the sign is +. Therefore two of the terms in the sum making up the determinant are

$$a_{1,1}a_{2,2}a_{3,3}a_{4,4} \qquad a_{1,3}a_{2,1}a_{3,2}a_{4,4}\ .$$

There are $4! = 24$ in all.

For 2×2 we get the list

Sequence	Inversions	Sign	Term
1 2		+	$a_{1,1}a_{2,2}$
2 1	(2, 1)	−	$-a_{2,1}a_{2,1}$

and for 3×3 we get the following list:

1 2 3		+	$a_{1,1}a_{2,2}a_{3,3}$
1 3 2	(3, 2)	−	$-a_{1,1}a_{2,3}a_{3,2}$
2 1 3	(2, 1)	−	
2 3 1	(2, 1) (3, 1)	+	
3 1 2	(3, 1) (3, 2)	+	
3 2 1	(3, 1) (3, 2) (2, 1)	−	

Exercise. Fill in the last column of the previous table.

Exercise. Make a table of all 24 sequences of 4 numbers like the one above (but do not fill in the last column).

Determinants and elementary row operations

We can use the definition given above to verify the facts mentioned earlier, and a few related ones.

(1) *If we interchange any two rows (or columns) of a matrix, we change the sign of the its determinant.*

Because the number of inversions changes from even to odd and vice-versa.

(2) *If we multiply a row (or column) by a constant c we multiply its determinant by c.*

Because each term in the determinant contains exactly one item from that row or column.

(3) *If two rows or columns are the same, the determinant is 0.*

Because there will be matching terms which cancel out.

(4) *If we replace row (or column) entries $a_{i,j}$ in one row by $b_{i,j} + c_{i,j}$ then the determinant is a sum of two determinants, one with the $b_{i,j}$ and one with the $c_{i,j}$.*

Because each term in the determinant will have a term $\cdots(b_{i,j}+c_{i,j})\cdots$ which splits into a sum of two terms $\cdots b_{i,j}\cdots$ and $\cdots c_{i,j}\cdots$.

(5) *If we add a multiple of one row (or column) to another, we don't change the determinant at all.*

This follows from (2), (3), and (4).

(6) *The determinant of a square matrix with all zeroes below (or above) its diagonal is the product of the diagonal entries.*

Because all but the diagonal term contains a 0.

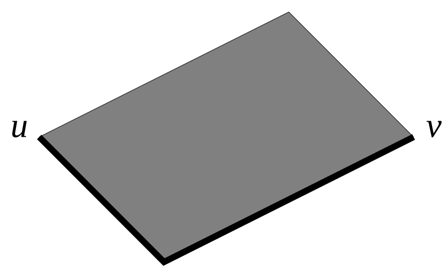
Determinants and volumes (again)

Any pair of vectors u, v in the plane, not lying along the same line, give rise to a parallelogram. This parallelogram has an **orientation** $\pm$ depending on whether u is rotated positively or negatively to get to the direction of v.

For example, the pair

$$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

gives this parallelogram



which has orientation $-$. The **oriented area** of such a parallelogram is the product of its orientation and its usual area.

If v_1 and v_2 do lie along the same line then they span a degenerate parallelogram which has volume 0.

Theorem. *If u and v are a pair of vectors. then the oriented area of the corresponding oriented parallelogram is the determinant of the matrix whose columns are u and v.*

This is because we can use row reduction to calculate both the oriented volume and the determinant.

(1) *If we swap the two columns we change the orientation and the sign of the determinant.*

(2) *If we multiply either of the vectors by a constant c we multiply both the determinant and the oriented area by c (even if $c < 0$.*

This operation just scales the parallelogram along one of its sides.

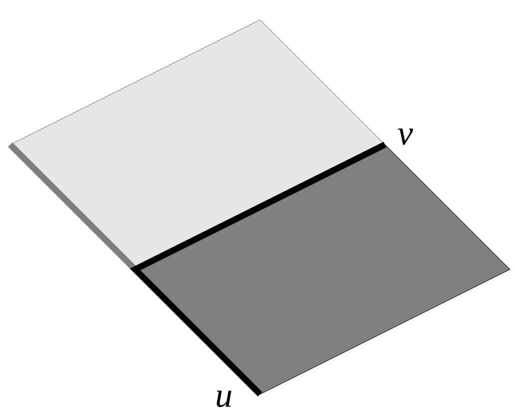
Example. In column reducing

$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

we first multiply column one by -1. The matrix becomes

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

u is relaced by $-u$, and the figure changes to



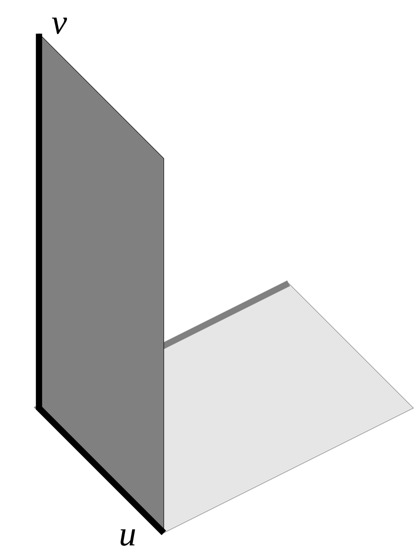
(3) *If we subtract a multiple of one column from another we do not change either the area or the determinant.*

We just slide one side of the parallelogram parallel to itself, or **shear** the parallelogram (like sliding a deck of cards).

Example continued. We next subtract twice the first column from the second. The matrix becomes

$$\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$$

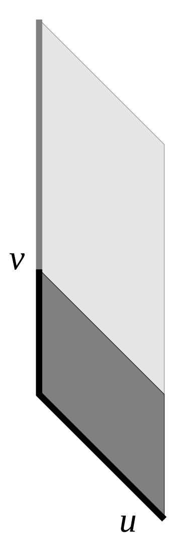
v is replaced by $v - 2u$, and the figure becomes



We then multiply the second column by 1/3. The area gets multiplied by 1/3 also. The matrix becomes

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

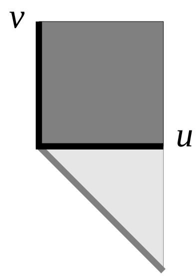
v is replaced by $v/3$, and the figure becomes



Fnally we add the second column to the first. The area remains the same. The matrix becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

u is replaced by $u + v$, and the figure becomes



Both the determinant and the area are equal to 1.

In other words, *we have performed a sequence of operations on the matrix, and at each stage the area and the determinant changed in the same way. At the end the two are equal. Therefore they were the same to begin with.*

The same thing happens in three dimensions: *The determinant of a* 3×3 *matrix is the oriented volume of the parallelopiped spanned by its columns.*

In three dimensions the orientation is determined by the right hand rule. That is to say, a triple u, v, w has orientation $+$ if they can be matched by this rule, and $-$ if they can be matched by the left hand rule. The effect of reflection in a plane, for example, reverses orientation.

Practical calculation

The number $n!$ grows very rapidly with n, and using the definition directly is an impractical way to calculate determinants. The effect of row operations on determinants suggests abetter way. In effect, we use Gauss elimination to write $WM = LU$, then get $\det M = \det W^{-1} \det L \det U$. The determinant of W is ± 1, whereas that of L is 1, and the determinant of U is the product of its diagonal elements.