Mathematics 307—December 6, 1995

Finding the matrices of linear transformations

I recall here for quick reference:

Proposition. *Suppose that T is a linear transformation, E and F bases. Let M_E be the matrix of T in the E-coordinate system, M_F that in the F-coordinate system. If $F = E\,A$ expresses the relationship between the two coordinate systems (or in other words the matrix A has as its columns the vectors in F expressed in E-coordinates) then*

$$M_F = A^{-1} M_E A, \quad M_E = A M_F A^{-1} .$$

This can be used in any one of several ways. One I want to demonstrate here is *how to find the matrix of a linear transformation described in geometrical terms.*

Example. Suppose we are working with the usual (x, y) plane. Let T be mirror reflection in the line $y = 2x$. What is the matrix of T in (x, y) coordinates?

The pattern will be the same in most cases. Step (1) We choose a coordinate system F in which T is simple, so that it is easy to calculate M_F. Step (2) We find the relationship between F and the coordinate system we are actually interested in. Step (3) Apply the Proposition.

Step (1) Here we choose f_1 along the line $y = 2x$, say $f_1 = (1, 2)$. Then we choose f_2 to be any vector perpendicular to f_2. In two dimensions it is simple to do this, because we know that (x, y) rotated by $90°$ is $(-y, x)$. So we set $f_2 = (-2, 1)$. The transformation takes f_1 to itself and flips f_2 into $-f_2$. So in the F-coordinate system we have

$$M_F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Step (2) The matrix A has the f_i as its columns. So

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad A^{-1} = \frac{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}{5} .$$

Step (3)

$$M_E = A M_F A^{-1} = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} .$$

Example. We now look at an example in three dimensions. The only reason for looking only at two dimensional stuff so far was to build up geometrical intuition, and everything I have said so far except the classification of linear transformations as scale changes, rotations, or shears is valid in $3D$ also.

Let T be projection parallel to the axis through $(1, 1, 1)$ onto the plane P perpendicular to it which passes through the origin. Here we choose f_1 to be $(1, 1, 1)$ itself. We want to choose f_2 and f_3 in the plane P. Now two vectors are perpendicular when their **dot product** is equal to 0. The dot product of (x, y, z) with $(1, 1, 1)$ is $x + y + z$, so the equation of the plane P is $x + y + z = 0$. We can find at least one vector in this plane just by guessing, say $f_2 = (1, -1, 0)$. We could choose f_3 to be independent vector there, say $(1, 0, -1)$, but as we see there is some advantage in similar problems to having f_3 perpendicular to both f_1 and f_2. In three dimensions we can do this by using the **cross product**.