Mathematics 307—September 25, 1995

Classification in higher dimensions

In the roughest classification, ignoring the distinction between real and complex numbers, each 2×2 matrix is similar to just one of two types

$$\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 1 \\ & \lambda_1 \end{bmatrix}$$

where it is possible that $\lambda_1 = \lambda_2$. Each 3×3 matrix is similar to one of these:

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}$$

The simplest way to summarize this is to say that each matrix is similar to one which is a sum of certain blocks of various sizes along the diagonal. For each dimension there is one kind of $m \times m$ block which has λ down the diagonal, 1 all along just above the diagonal, and all zeroes everywhere else. For $m = 4$, for example, the 4×4 block is this:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

This classification persists in all dimensions. I will not discuss here how these blocks can be calculated for any given matrix, but it is not much more difficult than it is for dimensions 2 and 3.

Exponential polynomials

There is a curious example where these blocks arise. Suppose we consider the vector space of all functions of the form $P(x)e^{\lambda x}$ where $P(x)$ is a polynomial of degree at most 2, where λ is some constant. It has the basis

$$\begin{aligned} e_1 &= e^{\lambda x} \\ e_2 &= x\, e^{\lambda x} \\ e_3 &= x^2 e^{\lambda x} \, . \end{aligned}$$

Let T be the operator taking $f(x)$ to $f'(x)$. It is linear since

$$(cf)' = cf', \quad (f+g)' = f' + g' \, .$$

Since

$$\begin{aligned} Te^{\lambda x} &= \lambda e^{\lambda x} \\ Txe^{\lambda x} &= \lambda x e^{\lambda x} + e^{\lambda x} \\ Tx^2 e^{\lambda x} &= \lambda x^2 e^{\lambda x} + 2x e^{\lambda x} \end{aligned}$$

the matrix of T with respect to this basis is

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{bmatrix}$$

If we change our basis so that

$$e_3 = \frac{x^2}{2}\,e^{\lambda x}$$

then the matrix is

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} .$$

Similarly, if we look at functions $P(x)e^{\lambda x}$ where P is allowed to have degree at most 3, and choose as basis

$$\begin{aligned} e_1 &= e^{\lambda x} \\ e_2 &= xe^{\lambda x} \\ e_3 &= (x^2/2)e^{\lambda x} \\ e_4 &= (x^3/6)e^{\lambda x} \end{aligned}$$

the matrix of differentiation is

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} .$$

This works for all degrees if we take a basis of functions

$$\frac{x^n}{n!}\,e^{\lambda x} .$$

The matrices we get can be applied to find formulas for the solutions of differential equations like

$$y'' + ay' + by = P(x)e^{\lambda x}$$

where $P(x)$ is a polynomial, a and $b$constants. The first thing to consider is the **homogeneous equation**

$$y'' + ay' + by = 0$$

Its set of solutions form a vector space, which is of dimension two since a solution is determined precisely by the numbers $y(0)$ and $y'(0)$. Since a and b are constant, if y is a solution then so is y'. Thus we have the linear operator $D\colon y \mapsto y'$ acting on the two dimensional space of solutions. There must exist at least one eigenvector, which will be an exponential function since these are the only functions taken into multiples of themselves by differentiation. We can find the exponent as a root of the polynomial equation

$$\lambda^2 + a\lambda + b = 0$$

There will be a problem when this equation has two roots. In this case the matrix of D is similar to

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

and the second solution we want satisfies the equation

$$y' = \lambda y + e^{\lambda x} .$$

This will be $xe^{\lambda x}$ is a second solution.

For the **inhomogeneous** equation, the simplest case is just the equation

$$y' = x^n e^{\lambda x}$$

which means we are looking for the integral

$$\int x^n e^{\lambda x}\, dx$$

where $\lambda \neq 0$. The point is that integration is the inverse of differentiation. The matrix describing differentiation on the spaces above is invertible as long as $\lambda \neq 0$. Therefore for any polynomial $P(x)$ there is a unique polynomial $Q(x)$, of the same degree as $P(x)$, such that

$$\frac{dQ(x)e^{\lambda x}}{dx} = P(x)e^{\lambda x}$$

if $\lambda \neq 0$. How can we find it explicitly? We can write the matrix

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \lambda I + N$$

where I is the 4×4 identity matrix and

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

note that

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can also write the product as

$$\lambda \left(I + \frac{N}{\lambda} \right)$$

and its inverse as

$$\frac{1}{\lambda} \left(I + \frac{N}{\lambda} \right)^{-1}$$

and evaluate the second expression just as we could with numbers

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{\lambda} \left(I + \frac{N}{\lambda} \right)^{-1} = \left(\frac{I}{\lambda} - \frac{N}{\lambda^2} + \frac{N^2}{\lambda^3} - \frac{N^3}{\lambda^4} \right)$$

since $N^4 = N^5 = \cdots = 0$.

Exercise. *How does differentiation act on the space of functions of the form*

$$a \cos \omega x + b \sin \omega x?$$

Choose a basis and write down the matrix.

Exercise. *How does differentiation act on the space of polynomials of degree at most n? Choose a basis and write down the matrix.*

Exercise. *Find a formula for*

$$\int x^n e^{cx}\,dx$$

by this method.

Exercise. *Let T be the linear operator*

$$Tf = f'' + f$$

acing on the space of functions $P(x)e^{-x}$ where $P(x)$ has degree at most 4. What is its matrix?

Exercise. *There exists a unique solution of the form $P(x)e^{-x}$ of the differential equation*

$$y'' + y = x^4 e^{-x}$$

Find it by this method, considering the operator $y \mapsto y'' + y$ as a linear operator.

Exercise. *How does $T : f \mapsto f' + f$ act on the space $P(x)e^{-x}$ with P of degree at most 3? Choose a basis and write down the matrix.*