Mathematics 307—December 6, 1995

Generalized eigenvalues and conservative systems

Assume M positive definite. To solve

$$Mx'' + Kx = 0$$

we convert it into a first order system by setting

$$y = Mx$$

and getting

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & M^{-1} \\ -K & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$v' = Av, \quad v = e^{At} v_0$$

We apply Gauss elimination to factor

$$M = L\,{}^tL, \quad M^{-1} = {}^tL^{-1}\,L^{-1}, \quad {}^tL\,M^{-1}L = I$$

and then multiply

$$\begin{bmatrix} {}^tL & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} 0 & M^{-1} \\ -K & 0 \end{bmatrix} \begin{bmatrix} {}^tL^{-1} & 0 \\ 0 & L \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L^{-1}K\,{}^tL^{-1} & 0 \end{bmatrix}$$

The matrix

$$K_* = L^{-1}K\,{}^tL^{-1}$$

is still symmetric. We can find an orthogonal X such that

$$X^{-1}K_*X = D$$

where D is diagonal. So

$$\begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -K_* & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} 0 & I \\ -D & 0 \end{bmatrix}$$

Finally we use

$$\begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

if $\omega = \sqrt{k}$. So we set

$$\Omega = \begin{bmatrix} \omega_1 & 0 & 0 & \cdots \\ 0 & \omega_2 & 0 & \cdots \\ 0 & 0 & \omega_3 & \cdots \\ \cdots & & & \end{bmatrix}$$

and get

$$\begin{bmatrix} \Omega & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -D & 0 \end{bmatrix} \begin{bmatrix} \Omega^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} = C$$

Thus if

$$\begin{aligned} Y &= \begin{bmatrix} \Omega & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix} \begin{bmatrix} {}^tL & 0 \\ 0 & L^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \Omega X^{-1}\,{}^tL & 0 \\ 0 & X^{-1}L^{-1} \end{bmatrix} \\ Y^{-1} &= \begin{bmatrix} {}^tL^{-1}X\Omega^{-1} & 0 \\ 0 & LX \end{bmatrix} \end{aligned}$$

then

$$Y A Y^{-1} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}, \quad A = Y^{-1} A Y, \quad e^{At} = Y^{-1} e^{Ct} Y$$

Finally

$$e^{Ct} = \begin{bmatrix} \cos\omega_i t & -\sin\omega_i t \\ \sin\omega_i t & \cos\omega_i t \end{bmatrix}$$

Lower triangular systems

We want to find x such that $Lx = c$. We find x_0, x_1 etc.

$$\begin{aligned} \ell_{0,0} x_0 &= c_0 \\ \ell_{i,0} x_0 + \ell_{i,1} x_1 + \cdots + \ell_{i,i} x_i &= c_i \\ x_0 &= c_0 / \ell_{0,0} \\ x_i &= c_i - \ell_{i,0} x_0 - \ell_{i,1} x_1 - \cdots \end{aligned}$$

We will call this with a parameter n, the dimension, so we can use partial vectors anmd matrices.

Cholesky factorization

We have

$$\begin{bmatrix} \lambda & 0 \\ \ell & \Lambda \end{bmatrix} \begin{bmatrix} \lambda & {}^t\ell \\ 0 & {}^t\Lambda \end{bmatrix} = \begin{bmatrix} \alpha & {}^ta \\ a & A \end{bmatrix}$$

which we solve by induction.

$$\begin{aligned} \lambda^2 &= \alpha \\ \lambda\,\ell &= a \\ \ell &= \lambda^{-1}\, a \\ \ell\,{}^t\ell + \Lambda\,{}^t\Lambda &= A \\ \Lambda\,{}^t\Lambda &= A - \ell\,{}^t\ell \end{aligned}$$

Note that $\ell\,{}^t\ell$ is a square matrix with entries $\ell_i \ell_j$.

For $i = 0$ to $n-2$
  $\lambda = \ell_{i,i} := \sqrt{m_{i,i}}$
  for $j = i+1$ to $n-1$
    $\ell_{j,i} := m_{j,i}/\lambda$
  for $j = i+1$ to $n-1$
    for $k = i+1$ to j
      $m_{j,k} := m_{j,k} - \ell_{j,i}\ell_{k,i}$
$\ell_{n-1,n-1} := \sqrt{m_{n-1,n-1}}$

Lower triangular inverses

To solve

$$\begin{bmatrix} \chi & 0 \\ x & X \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ \ell & \Lambda \end{bmatrix} = I$$

or

$$\chi\lambda = 1, \quad \lambda x + X\ell = 0$$

is the easiest way to do it inductively. Note that we should also have a routine for finding Lx where L is lower triangular.