Mathematics 307—October 4, 1995

Calculating moments

Suppose we are given a collection of point masses m_α at locations $\mathbf{r}^\alpha = (\mathbf{r}^\alpha_x, \mathbf{r}^\alpha_y, \mathbf{r}^\alpha_z)$. Assume they are all assembled together to make up a single object. The **centre of gravity** of the object is defined to be the average position of the point masses that make it up, weighted according to mass. It is the vector

$$\frac{\sum_\alpha m_\alpha \mathbf{r}^\alpha}{\sum m_\alpha}$$

The **moment of inertia matrix** of the object is the 3×3 matrix

$$\mathcal{I} = \begin{bmatrix} \sum m_\alpha(\mathbf{r}^\alpha_y \mathbf{r}^\alpha_y + \mathbf{r}^\alpha_z \mathbf{r}^\alpha_z) & -\sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_y & -\sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_z \\ -\sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_y & \sum m_\alpha(\mathbf{r}^\alpha_x \mathbf{r}^\alpha_x + \mathbf{r}^\alpha_z \mathbf{r}^\alpha_z) & -\sum m_\alpha \mathbf{r}^\alpha_y \mathbf{r}^\alpha_z \\ -\sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_z & -\sum m_\alpha \mathbf{r}^\alpha_y \mathbf{r}^\alpha_z & \sum m_\alpha(\mathbf{r}^\alpha_x \mathbf{r}^\alpha_x + \mathbf{r}^\alpha_y \mathbf{r}^\alpha_y) \end{bmatrix}$$

where the coordinate system is assumed to have their origin at the centre of gravity.

The **principal axes** of the object are the eigenvectors of the matrix $\mathcal{I}$. Physically, $\mathcal{I}$ is used to relate angular momentum L to angular velocity ω according to the recipe $L = \mathcal{I}\omega$. This is important because physical laws are best expressed in terms of momentum although the motion we see is directly related to velocity.

How can $\mathcal{I}$ be calculated for a given object? Suppose we are given the masses and their positions. One path we can follow is to calculate the centre of gravity, shift coordinates to make the c. g. the origin, and then calculate the matrix directly. This is somewhat clumsy, mostly because it is somewhat inflexible—adding another mass to the assembly forces us to do a great deal of calculation. Instead, I suggest working directly in terms of the original data and introducing the coordinate shift at the end.

Define the **moments** of the assembly to be

$$\begin{aligned} M &= \sum m_\alpha \\ M_x &= \sum m_\alpha \mathbf{r}^\alpha_x \\ M_y &= \sum m_\alpha \mathbf{r}^\alpha_y \\ M_z &= \sum m_\alpha \mathbf{r}^\alpha_z \\ M_{xx} &= \sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_x \\ M_{xy} &= \sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_y \\ M_{xz} &= \sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_z \\ M_{yy} &= \sum m_\alpha \mathbf{r}^\alpha_y \mathbf{r}^\alpha_y \\ M_{xz} &= \sum m_\alpha \mathbf{r}^\alpha_x \mathbf{r}^\alpha_z \\ M_{zz} &= \sum m_\alpha \mathbf{r}^\alpha_z \mathbf{r}^\alpha_z \\ M_{yx} &= M_{xy} \\ M_{zx} &= M_{xz} \\ M_{zy} &= M_{yz} \end{aligned}$$

The first moment M is just the total mass. The rest are weighted sums of functions of the coordinates. The centre of gravity thus has coordinates

$$(M_x/M, M_y/M, M_z/M)$$

and if the c. g. happens to be the origin, the matrix $\mathcal{I}$ is then

$$\begin{bmatrix} M_{yy}+M_{zz} & -M_{xy} & -M_{xz} \\ -M_{yx} & M_{xx}+M_{zz} & -M_{yz} \\ -M_{zx} & -M_{zy} & M_{xx}+M_{yy} \end{bmatrix}$$

Moments have the convenient property that if we assemble two objects together then the moments of the composite object are the sums of the moments of the two components.

The steps in calculating $\mathcal{I}$ are these: (1) Calculate the moments of the collection of masses. (2) Calculate the centre of gravity. Call it $\mu = (\mu_x, \mu_y, \mu_z)$. (3) Shift coordinates so the c. g. becomes the origin. A point with original coordinates (x, y, z) has new coordinates $(x - \mu_x, y - \mu_y, z - \mu_z)$. The moments in the shifted coordinates are

$$\begin{aligned}
M^* &= M \\
M^*_x &= \sum m_\alpha(\mathbf{r}^\alpha_x - \mu_x) \\
&= M_x - M\mu_x \\
&= 0 \\
M^*_y &= M_y - M\mu_y \\
&= 0 \\
M^*_z &= M_z - M\mu_z \\
&= 0 \\
M^*_{xx} &= \sum m_\alpha(\mathbf{r}^\alpha_x - \mu_x)(\mathbf{r}^\alpha_x - \mu_x) \\
&= M_{xx} - \mu_x M_x - \mu_x M_x + M\mu_x\mu_x \\
M^*_{xy} &= M_{xy} - \mu_y M_x - \mu_x M_y + M\mu_x\mu_y \\
M^*_{xz} &= M_{xz} - \mu_z M_x - \mu_x M_z + M\mu_x\mu_z \\
&\ldots
\end{aligned}$$

If the object we are considering is a continuous distribution of masses, the moments are integrals

$$\begin{aligned}
M &= \int dm \\
M_x &= \int x\, dm \\
M_{xx} &= \int x^2\, dm \\
&\ldots
\end{aligned}$$