Mathematics 307—October 11, 1995

Angular motion and matrices

Three dimensional geometry is somewhat complicated, and 3D motion—even of rigid structures—is even more complicated. Matrices help sort out the complications.

Angular velocity

Suppose that a point is rotating around an axis with a certain radial speed. *The* **angular velocity** ω *of the point is by convention turned into a vector according to the convention that the direction of the vector is along the axis, oriented according to the right hand rule, and its length is the radial speed of the point.* We shall perhaps understand why this is convenient a bit later; it is not an obviously useful idea. It makes at least some sense, however, because of this fact: *If the point is located at position* (x, y, z) *and*

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

then the actual velocity of the point is

$$\mathbf{v} = \omega \times \mathbf{r}\ .$$

Exercise. *If a particle with position vector* $\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ *is rotating clockwise around the axis* $x = y = z$ *(clockwise as seen looking from this vector towards the origin) with a speed of* 1^r *per second, what is its linear velocity?*

A priori all we know about the motion of a rigid body is that it doesn't change the shape or orientation of the body, which means that it is essentially described by a special orthogonal matrix. To be a little more precise, if we choose a very small unit of time Δt then the transformation moving the body in that interval will be a linear transformation not very different from the identity transformation, and it will be special orthogonal. We know, however, that any special orthogonal matrix is a rotation, and this means, if we take the limit as $\Delta t \to 0$, means that in fact *at any given instant the body is rotating around an axis*, which is called its **instantaneous axis**. This axis will generally change, however, as time proceeds.

Exercise. *Suppose that*

$$T = I + \Omega\Delta t + \text{ terms of order } \Delta t^2$$

is an orthogonal transformation for all t. *What condition must* Ω *satisfy? (Hint. Write out* ${}^tT\,T$*).*

Moments

If $\mathbf{v}$ is a vector applied at a position (x, y, z) and

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

Then the **moment** of $\mathbf{v}$ with respect to the origin is the cross product

$$\mathbf{r} \times \mathbf{v}\ .$$

If the particle has mass m is moving with velocity $\mathbf{v}$, then its (linear) momentum is the vector $m\mathbf{v}$, and its **angular momentum** with respect to the origin is the moment of its momentum, or in other words

$$\mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times (\omega \times \mathbf{r})$$

if ω is its angular velocity. This is the same as

$$m(\mathbf{r} \bullet \mathbf{r})\omega - m(\omega \bullet \mathbf{r})\mathbf{r} = m(\mathbf{r} \bullet \mathbf{r}) \left[\omega - \frac{\omega \bullet \mathbf{r}}{\mathbf{r} \bullet \mathbf{r}} \mathbf{r}\right]$$

which is the same as $m\|\mathbf{r}\|^2$ times the projection of ω onto the plane perpendicular to $\mathbf{r}$. It is also

$$m \begin{bmatrix} \mathbf{r}_y^2 + \mathbf{r}_z^2 & -\mathbf{r}_x \mathbf{r}_y & -\mathbf{r}_x \mathbf{r}_z \\ -\mathbf{r}_x \mathbf{r}_y & \mathbf{r}_x^2 + \mathbf{r}_z^2 & -\mathbf{r}_y \mathbf{r}_z \\ -\mathbf{r}_x \mathbf{r}_z & -\mathbf{r}_y \mathbf{r}_z & \mathbf{r}_x^2 + \mathbf{r}_y^2 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} .$$

The matrix

$$\mathcal{I}(m, \mathbf{r}) = m \begin{bmatrix} \mathbf{r}_y^2 + \mathbf{r}_z^2 & -\mathbf{r}_x \mathbf{r}_y & -\mathbf{r}_x \mathbf{r}_z \\ -\mathbf{r}_x \mathbf{r}_y & \mathbf{r}_x^2 + \mathbf{r}_z^2 & -\mathbf{r}_y \mathbf{r}_z \\ -\mathbf{r}_x \mathbf{r}_z & -\mathbf{r}_y \mathbf{r}_z & \mathbf{r}_x^2 + \mathbf{r}_z^2 \end{bmatrix}$$

is called the **moment of inertia matrix** of the particle with mass m at position $\mathbf{r}$.

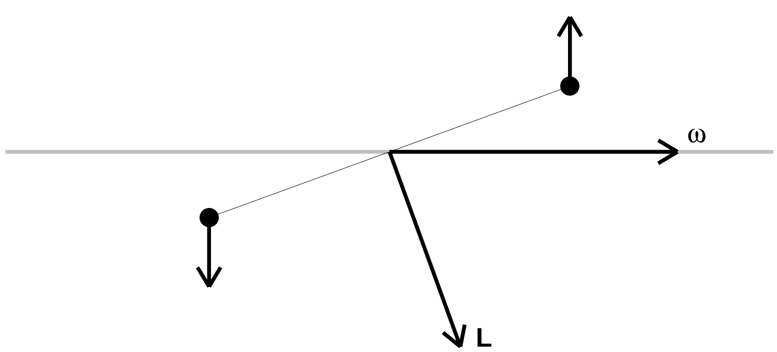
Angular momentum is difficult to understand. Let's consider some possible experiments in order to get a better feel for it.

Fix two weights of mass m_1 and m_2 at the end of a light but strong rod. Assume in fact that the rod itself weighs nothing, and that the masses are concentrated at the ends. Fix the rod on an axle, at an angle of φ, at some distance along the rod, and at the centre of the axle. Rotate the axle with angular velocity ω, which therefore points along the axle.

The centre of gravity of the rod will be at location (ℓ_1, ℓ_2) on the rod, where $\ell_1 + \ell_2$ is the total length of the rod, and

$$m_1 \ell_1 = m_2 \ell_2 .$$

If the rod is pinned down at an arbitrary point there may be forces tending to pull the whole axle out, but if it is pinned to the axle at its centre of gravity, then the vector sum of all centrifugal forces will be 0, and there will not be any overall pull on the system. This is not the same as saying these forces have no effect.



Unless the rod is placed perpendicularly to the axle, or right along it, centrifugal force will push the ends of the rod away from the axle. This will cause one end of the axle to be pulled out in one way, and the other in the opposite direction. This effect is called **torque**. We shall see in a later section that torque is related to angular momentum as force is related to the usual momentum. We shall see that the presence of torque means that the angular momentum must be changing. Since the angular velocity is not changing in this system, this indicates that the relationship bewteen angular velocity and angular momentum is more complicated than that between the usual velocity and the usual momentum.

Angular momentum is a vector. Its length is not changing, but we shall see that it is rotating with the rod to which the masses are attached.

The angular momentum of the system is the sum of the angular momenta of its different constituents. A mass m at position $\mathbf{r}$ contributes a term equal to the quantity $m\|\mathbf{r}\|^2$ multiplied by the projection of ω perpendicular to $\mathbf{r}$. If we have several objects in a structure, of mass m_i and location $\mathbf{r}_i$, the total angular momentum is therefore

$$\left[\sum \mathcal{I}(m_i, \mathbf{r}_i)\right] \omega \ .$$

The matrix sum

$$\mathcal{I} = \left[\sum \mathcal{I}(m_i, \mathbf{r}_i)\right]$$

is called the **moment of inertia matrix** of the structure. It is replaced by a matrix whose coefficients are integrals

$$\int (x^2 + y^2)\, dm, etc.$$

if the body is continuous.

The relationship between angular velocity and angular momentum is

$$\mathbf{L} = \mathcal{I}\omega \ .$$

The real point here is that *the angular momentum does not necessarily point in the same direction as the axis of revolution.*

Suppose we ask the question: *Under what circumstances will* $\mathbf{L}$*and* ω *point in the same direction?* This will happen when $\mathbf{L} = c\omega$ for some constant c, which amounts to the equation

$$\mathcal{I}\omega = c\omega \ .$$

This is just the equation satisfied by an eigenvector with eigenvalue ω, since it can be rewritten as

$$(\mathcal{I} - cI)\omega = 0 \ .$$

In other words, the direction where angular momentum and angular velocity point in the same direction are the eigenvector directions of the moment of inertia matrix $\mathcal{I}$. These are called the **principal axes** of the structure.

For explicit calculation:

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_{x,x} & \mathcal{I}_{x,y} & \mathcal{I}_{x,z} \\ \mathcal{I}_{y,x} & \mathcal{I}_{y,y} & \mathcal{I}_{y,z} \\ \mathcal{I}_{z,x} & \mathcal{I}_{z,y} & \mathcal{I}_{z,z} \end{bmatrix}$$

where

$$\begin{aligned} \mathcal{I}_{x,x} &= \sum m_i(y_i^2 + z_i^2) \\ \mathcal{I}_{y,y} &= \sum m_i(x_i^2 + z_i^2) \\ \mathcal{I}_{z,z} &= \sum m_i(x_i^2 + y_i^2) \\ \mathcal{I}_{x,y} = \mathcal{I}_{y,x} &= -\sum m_i x_i y_i \\ \mathcal{I}_{x,z} = \mathcal{I}_{z,x} &= -\sum m_i x_i z_i \\ \mathcal{I}_{y,z} = \mathcal{I}_{z,y} &= -\sum m_i y_i z_i \end{aligned}$$

The matrix $\mathcal{I}$ is symmetric. Its eigenvalues are always real, and in fact non-negative. The physics suggests also the following fact: the principal axes of a system are always perpendicular to each other if the eigenvalues are distinct, and may be always chosen to be mutually perpendicular. The matrix $\mathcal{I}$ depends on which coordinate system we calculate in. The simplest is that where the coordinate axes are the principal axes. In that case the matrix $\mathcal{I}$ is diagonal.

The eigenvalue of $\mathcal{I}$ corresponding to a principal axis $\mathbf{u}$ is the moment of inertia around the axis $\mathbf{u}$.

If we have a structure with principal normalized axes $\mathbf{u}_i$ and eigenvalues λ_i, we can calculate the angular momentum of the system in terms of angular velocity in the following fashion: resolve ω into components $\omega_i \mathbf{u}_i$ with respect to the principal axes, so that

$$\omega = \omega_1 \mathbf{u}_1 + \omega_2 \mathbf{u}_2 + \omega_3 \mathbf{u}_3 \ .$$

Then

$$\mathbf{L} = \lambda_1 \omega_1 \mathbf{u}_1 + \lambda_2 \omega_2 \mathbf{u}_2 + \lambda_3 \omega_3 \mathbf{u}_3 \ .$$

This is as close as we can come to understanding intuitively how angular momentum and angular velocity are related. Note at any rate that the angular momentum moves with the structure, even if the axis of revolution doesn't change.

Exercise. *(a) Find the centre of gravity of a tennis racket. Assume it is constructed by adding a circle of radius* 10 *cm to a thin handle of length* 20 *cm, and that the linear density is* 1 *gm/cm around the circle,* 2 *gm/cm in the handle. This calculation will use the sum of two integrals, one over each component.*

(b) Find its moment of inertia matrix $\mathcal{I}$ with respect to its centre of gravity—its principal axes (clear) and eigenvalues.

Exercise. *Do the same for a system made up of three objects: (i) mass* 3*, location* $-\mathbf{i}$*; (ii) mass* 1*, location* $\mathbf{i} + \mathbf{j}$*; (iii) mass* 2*, location* $\mathbf{i} - \mathbf{j}$.

Free rotational motion

Torque is the moment of force

$$\tau = \mathbf{r} \times \mathbf{F} \ .$$

Newton's Law $F = ma$ can be expressed also as

$$F = \frac{d\, mv}{dt} \ .$$

and for three dimensional systems it can be supplemented by the equation

$$\tau = \frac{d\mathbf{L}}{dt}$$

which follows from it. We won't have to calculate torques explicitly, but the consequence we need is that *torque vanishes if and only if angular momentum is constant.* If we throw a tennis racket into the air, for example, force of gravity exerts no torque (neglecting the tiny torque exerted by the tiny change in gravitational force with height). The centre of gravity of the racket follows a parabolic arc, and the racket spins if it is started out spinning. But if the original spin is not along one of the principal axes, the axis of revolution changes precisely because the angular momentum does not.

Summary

A rigid structure moving in three dimensions has at any moment two minds of motion—**translational** and **rotational**. The translational motion is controlled by the vector sum of forces acting on it, and the rotational motion is controlled by the total torque τ of the forces. The translational effect of forces is through Newton's Law

$$F = ma = m\frac{dv}{dt} = \frac{d\,mv}{dt}$$

which is relatively simple to understand, basically because it may be considered to act on the centre of mass of the structure. The effect of the torque is to move the structure in a more complicated way. *Torque acts directly on the angular momentum* $\mathbf{L}$, *and this is the reason angular momentum is an important concept.* Explicitly

$$\tau = \frac{d\mathbf{L}}{dt} \, .$$

Angular momentum is not a simple concept to understand geometrically, however. The concept which is relatively simple to understand is angular velocity ω, and the angular momentum and the angular velocity are related through the moment of inertia matrix

$$\mathbf{L} = \mathcal{I}\omega \, .$$

This relationship is simplest to understand if coordinates are chosen inside the structure in terms of the principal axes, which are the directions of the eigenvectors of the structure.

Knowing what the angular velocity is at any moment will allow you to describe completely the rotational component of a structure's motion. The transition from knowing velocity to finding position amounts to solving a certain differential equation; it amounts to integrating the angular velocity, but in a slightly tricky way. What really complicates things is the somewhat circular fact that the torque on a structure is usually a function of its position! This, however, is similar to the simpler theory of moving point particles in complicated force fields.