Mathematics 307—October 23, 1995

Mid-term solutions

1. *Classify the following matrices by writing out the standard matrices they are similar to. In the case of 3D rotations, find the axis of rotation. In the case of 2D scale changes, draw the lines of eigenvectors.*

$$(a) \begin{bmatrix} 2.64 & -0.48 \\ -0.48 & 2.36 \end{bmatrix}$$
$$(b) \begin{bmatrix} 5 & -4 \\ 9 & -7 \end{bmatrix}$$
$$(c) \begin{bmatrix} 3 & 7 & -3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$
$$(d) \begin{bmatrix} 0.899519 & -0.404006 & -0.166266 \\ 0.345994 & 0.891146 & -0.293510 \\ 0.266747 & 0.206490 & 0.941386 \end{bmatrix}$$
$$(e) \begin{bmatrix} 6 & -1 & -9 \\ -9 & 2 & 15 \\ 5 & 1 & -4 \end{bmatrix}$$

(a) The characteristic polynomial is $\lambda^2 - 5\lambda + 6$ and the matrix is therefore similar to the scale change

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

The eigenvectors for 2 are the line

$$0.64x - 0.48y = 0$$

which goes through $(3, 4)$. The other eigenvectors lie on the line perpendicular to this one.

(b) The characteristic polynomial is $\lambda^2 + 2\lambda + 1$. The matrix is not a scalar matrix, so it is similar to the generalized shear

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

(c) The characteristic polynomial is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2$. The rank of $A - 2I$ is two, so the matrix is similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

(d) This is a special orthogonal matrix, therefore a rotation. The axis is made up of eigenvectors for 1, which are multiplies of $(0.500, -0.433, 0.750)$. The trace of the matrix is 2.732. The matrix is similar to some rotation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

It must have the same characteristic polynomial as any similar matrix, so we have

$$2.732 = 1 + 2\cos\theta$$

if θ is the angle of rotation. So $\theta = \pm 30°$.

(e) Characteristic polynomial $\lambda^3 - 4\lambda^2 + \lambda + 6$, similar to a diagonal matrix with entries -1, 2, 3.

2. *(a) What is the perpendicular projection of $(1, 1, -1)$ onto the plane $x - y + 2z = 0$?*

(b) What is the perpendicular reflection of $(1, 1, -1)$ through the plane $x - y + 2z = 0$?

(c) What is the matrix of perpendicular reflection through the plane $x - y + 2z = 0$?

Let $u = (1, 1, -1)$, $v = (1, -1, 2)$.

(a) The projection of u onto the line through v is

$$u_0 = \frac{u \bullet v}{v \bullet v} v = (-1/3, 1/3, -2/3)$$

Its complementary projection is

$$u_\perp = u - u_0 = (1, 1, -1) - (-1/3, 1/3, -2/3) = (4/3, 2/3, -1/3)$$

(b) Its reflection is

$$u - 2u_0 = u_\perp - u_0 = (4/3, 2/3, -1/3) - (-1/3, 1/3, -2/3) = (5/3, 1/3, 1/3)$$

(c) The fastest way to compute the matrix is to apply the same procedure to **i**, **j**, **k**. We get

$$\frac{1}{6}\begin{bmatrix} 4 & 2 & -4 \\ 2 & 4 & 4 \\ -4 & 4 & -2 \end{bmatrix} .$$

We can also use the formula for the matrix of perpendicular projection to find the formula here to be

$$\frac{1}{||v||^2}\begin{bmatrix} -v_x^2 + v_y^2 + v_z^2 & -2v_x v_y & -2v_x v_z \\ -2v_x v_y & v_x^2 - v_y^2 + v_z^2 & -2v_y v_z \\ -2v_x v_z & -2v_y v_z & v_x^2 + v_y^2 - v_z^2 \end{bmatrix}$$

3. *In the (x, y, z) coordinate system, the matrix of a certain linear transformation is*

$$M = \begin{bmatrix} -2 & -1 & -2 \\ -3 & 0 & -2 \\ 6 & 2 & 5 \end{bmatrix}$$

Now make up a new basis from the vectors whose coordinates in the (x, y, z) system are $(0, -2, 1)$, $(2, 0, -3)$. $(0, 0, 1)$.

(a) What is the matrix of T in this system?

(b) Use the previous result to find a matrix X such that $X^{-1}MX$ is one of the standard forms, and write down that form.

Call u_1, u_2, u_3 the vectors of the new basis F.

Thus

$$F = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 1 & -3 & 1 \end{bmatrix}, \quad F^{-1} = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 3/2 & 1/2 & 1 \end{bmatrix}$$

since the determinant of F is 4. The new matrix is $F^{-1}M_E F$ which is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

This is very near a standard form. It says that

$$\begin{aligned} Tu_1 &= u_1 \\ Tu_2 &= u_2 \\ Tu_3 &= u_3 - u_2 + u_1 \end{aligned}$$

and a slight change of basis $v_1 = u_1$, $v_2 = -u_2 + u_1$, $v_3 = u_3$ makes it into the standard form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

4. *Let T be the linear operator taking f to f' acting on the space of functions of the form $P(x)\cos\omega x + Q(x)\sin\omega x$ where P and Q are polynomials of degree at most one.*

(a) Choose a basis for this vector space, and write down the matrix of T.

(b) Find its eigenvalues and eigenvectors.

Differentiation takes

$$\begin{aligned} \cos\omega x &\mapsto -\omega\sin\omega x \\ \sin\omega x &\mapsto \omega\cos\omega x \\ x\cos\omega x &\mapsto -\omega\sin\omega x + \cos\omega x \\ x\sin\omega x &\mapsto \omega\cos\omega x + \sin\omega x \end{aligned}$$

and therefore the matrix is

$$M = \begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}$$

Then

$$M - \lambda I = \begin{bmatrix} -\lambda & \omega & 1 & 0 \\ -\omega & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & \omega \\ 0 & 0 & -\omega & -\lambda \end{bmatrix}$$

Its determinant is the square of that of

$$\begin{bmatrix} \lambda & \omega \\ -\omega & -\lambda \end{bmatrix}$$

which is $(\lambda + \omega^2)$. Its eigenvalues are $\lambda = \pm i\omega$, repeated twice. The matrix $M - \lambda I$ has rank 3, and the eigenvectors are the functions $\cos\omega x \mp i\sin\omega x$.

5. *An object is assembled from three masses $m_1 = 1$, $m_2 = 3$, $m_3 = 2$. They are located respectively at coordinates*

$$\begin{aligned} (x_1, y_1, z_1) &= (-1, \ \ 1, \ \ 0) \\ (x_2, y_2, z_2) &= (\ \ 1, \ \ 1, \ \ 0) \\ (x_3, y_3, z_3) &= (\ \ 1, \ \ 2, \ \ 0) \end{aligned}$$

Find (a) the centre of gravity of the object and (b) its principal axes through its centre of gravity.

All moments with a z factor vanish. For the others we make up a table

m	x	y	mx	my	mx^2	mxy	my^2
1	-1	1	-1	1	1	-1	1
3	1	1	3	3	3	3	3
2	1	2	2	4	2	4	8
M			M_x	M_y	M_{xx}	M_{xy}	M_{yy}
6			4	8	6	6	12

(a) The centre of gravity is

$$(M_x/M, M_y/M, 0) = (2/3, 4/3, 0)$$

(b) The moments around the c.g. are

$$\begin{aligned} M^*_{xx} &= M_{xx} - M_x^2/M \\ &= 10/3 \\ M^*_{xy} &= M_{xy} - M_x M_y/M \\ &= 2/3 \\ M^*_{yy} &= M_{yy} - M_y^2/M \\ &= 4/3 \end{aligned}$$

The matrix $\mathcal{I}$ is therefore

$$\begin{bmatrix} M^*_{yy} & -M^*_{xy} & 0 \\ -M^*_{xy} & M^*_{xx} & 0 \\ 0 & 0 & M^*_{xx} + M^*_{yy} \end{bmatrix}$$

Its eigenvectors are the z-axis and the eigenvectors for the matrix

$$\begin{bmatrix} M^*_{yy} & -M^*_{xy} \\ -M^*_{xy} & M^*_{xx} \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}$$