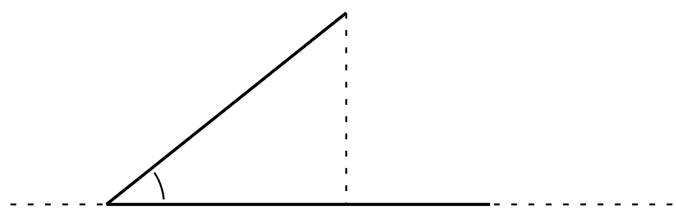
Mathematics 307—October 11, 1995

Projections

The **projection** of a vector $\mathbf{u}$ onto a line ℓ is the vector you get by dropping a perpendicular from the head of $\mathbf{u}$ onto ℓ. If we are given a vector $\mathbf{v}$ pointing in the direction of that line, the projection will be a scalar multiple of $\mathbf{v}$. The **signed length** of the projection is

$$||\mathbf{u}|| \cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{||\mathbf{v}||} \,.$$



It will be positive if the projection is on the same side as $\mathbf{v}$, zero if the projection vanishes, otherwise negative. In other words:

The projection of $\mathbf{u}$ *onto the line through* $\mathbf{v}$ *is the same as the vector*

$$\frac{\mathbf{u} \bullet \mathbf{v}}{||\mathbf{v}||} \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v} \,.$$

Now let P be the plane perpendicular to the vector $\mathbf{v}$. Any vector $\mathbf{u}$ can be expressed as the sum of its two components, one parallel to $\mathbf{v}$ and one perpendicular to it. The perpendicular component is the perpendicular projection of $\mathbf{u}$ onto the plane P. The formula for it is therefore

$$\mathbf{u} - \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v}$$

Let T be the linear transformation taking a vector to its perpendicular projection along the line through $\mathbf{u}$. Its matrix has as its columns the images of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. It is therefore

$$\frac{1}{||\mathbf{v}||^2} \begin{bmatrix} \mathbf{v}_x\mathbf{v}_x & \mathbf{v}_y\mathbf{v}_x & \mathbf{v}_z\mathbf{v}_x \\ \mathbf{v}_x\mathbf{v}_y & \mathbf{v}_y\mathbf{v}_y & \mathbf{v}_z\mathbf{v}_y \\ \mathbf{v}_x\mathbf{v}_z & \mathbf{v}_y\mathbf{v}_z & \mathbf{v}_z\mathbf{v}_z \end{bmatrix}$$

and the matrix of the complementary projection is

$$\begin{aligned} & I - \frac{1}{||\mathbf{v}||^2} \begin{bmatrix} \mathbf{v}_x\mathbf{v}_x & \mathbf{v}_y\mathbf{v}_x & \mathbf{v}_z\mathbf{v}_x \\ \mathbf{v}_x\mathbf{v}_y & \mathbf{v}_y\mathbf{v}_y & \mathbf{v}_z\mathbf{v}_y \\ \mathbf{v}_x\mathbf{v}_z & \mathbf{v}_y\mathbf{v}_z & \mathbf{v}_z\mathbf{v}_z \end{bmatrix} \\ &= \frac{1}{||\mathbf{v}||^2} \begin{bmatrix} \mathbf{v}_x\mathbf{v}_x + \mathbf{v}_y\mathbf{v}_y + \mathbf{v}_z\mathbf{v}_z & 0 & 0 \\ 0 & \mathbf{v}_x\mathbf{v}_x + \mathbf{v}_y\mathbf{v}_y + \mathbf{v}_z\mathbf{v}_z & 0 \\ 0 & 0 & \mathbf{v}_x\mathbf{v}_x + \mathbf{v}_y\mathbf{v}_y + \mathbf{v}_z\mathbf{v}_z \end{bmatrix} - \frac{1}{||\mathbf{v}||^2} \begin{bmatrix} \mathbf{v}_x\mathbf{v}_x & \mathbf{v}_y\mathbf{v}_x & \mathbf{v}_z\mathbf{v}_x \\ \mathbf{v}_x\mathbf{v}_y & \mathbf{v}_y\mathbf{v}_y & \mathbf{v}_z\mathbf{v}_y \\ \mathbf{v}_x\mathbf{v}_z & \mathbf{v}_y\mathbf{v}_z & \mathbf{v}_z\mathbf{v}_z \end{bmatrix} \\ &= \frac{1}{||\mathbf{v}||^2} \begin{bmatrix} \mathbf{v}_y\mathbf{v}_y + \mathbf{v}_z\mathbf{v}_z & -\mathbf{v}_y\mathbf{v}_x & -\mathbf{v}_z\mathbf{v}_x \\ -\mathbf{v}_x\mathbf{v}_y & \mathbf{v}_x\mathbf{v}_x + \mathbf{v}_z\mathbf{v}_z & -\mathbf{v}_z\mathbf{v}_y \\ -\mathbf{v}_x\mathbf{v}_z & -\mathbf{v}_y\mathbf{v}_z & \mathbf{v}_x\mathbf{v}_x + \mathbf{v}_y\mathbf{v}_y \end{bmatrix} \end{aligned}$$

There is one other formula involving projections which we shall need later.

Proposition. *For any 3D vectors u and v*

$$u \times (v \times u)$$

is equal to $\|u\|^2$ times the projection of v onto the plane perpendicular to u.

For the proof, we may as well divide by $\|u\|^2$, and can assume that $\|u\| = 1$. If v has the same direction as u then $u \times (v \times u)$ vanishes, as does the projection. If v is perpendicular to u then $v \times u$ is equal to v rotated by -90° around u, and u crossed with this in turn rotates it back to v. Thus also agrees with the projection. Since both the expression $u \times (v \times u)$ and the projection of v are linear in v, this proves the claim.

This also follows from the more general formula

$$u \times (v \times w) = (u \bullet w)v - (u \bullet v)w \ .$$

The triple product is perpendicular to both u and to $v \times w$. Since it is perpendicular to $v \times w$ it must lie in the plane containing v and w, hence will be a linear combination of v and w. The formula just finds the coefficients of this linear combination explicitly. To prove it, write v as a sum of two components v_0 and $v_\perp$, the first parallel to w and the second perpendicular to it. It suffices to deal with each component separately. For $v_0 = cw$ the formula asserts that $0 = 0$. Thus we may as well assume v is perpendicular to w. We amy also divide both sides by $\|v\|$ and $\|w\|$ and hence may assume v and w to be of length 1 as well as perpendicular. In that case $v \times w$ also has length 1 and makes up a rectangular frame together with v and w. We may choose these three vectors as basis. In this case the calculation is very simple.